

An Adaptive Stencil Linear Deviation Method for Wave Equations

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ABSTRACT

Wave Equations are partial differential equations (PDEs) which are used to model nearly all finite speed wave transmissions, such as sound and radar. The PDEs describe how wave information propagates in time by modeling displacement from equilibrium. Ideally, the exact position of the wave is known at some point in time, and the position of the wave at future times needs to be determined. In practice, however, the exact position may only be known at a finite number of points in space. Therefore, numerical approximations are needed so that the position of the wave at a future time can be predicted. This research focuses on the numerical methods used for these approximations.

THE WAVE EQUATION

One example of a hyperbolic partial differential equation (PDE) is the acoustic wave equation. This PDE models a three dimensional wave which may travel in any direction. In [4], it is shown that the larger acoustic problem can be approximated by a collection of one-dimensional, one-way wave equations which have the form

$$u_t + cu_x = 0 \quad u(x, 0) = f(x). \quad (1)$$

In this equation, c is the wave speed and $u(x, t)$ measures the displacement of the wave from equilibrium at each point in space x and at each time t . This equation describes how the wave changes from one point in time to the next. Initially ($t=0$), the exact position of the wave is given by $f(x)$, and the position of the wave at future times ($t>0$) must be determined. One added complication is that, in practice, only a finite amount of information is known initially (the value of u at a limited number of points).

METHODS

To approximate the value of the solution u at points in time after the initial time, the Taylor Series expansion [2,3,5] is used,

$$u(x, t + \Delta t) = u(x, t) + \Delta t u_x(x, t) + \frac{(\Delta t)^2}{2!} u_{xx}(x, t) + \dots \quad (2)$$

This implies that the value of a function at a later point in time may be determined, provided the derivative values are known at time t . Since the initial function is not known for all values, only a finite number of data points in space, the derivatives will need to be approximated.

The initial data provides information in space, but at only a single time ($t=0$), making time derivative approximations difficult. From the PDE, (1), we know $u_t = -u_x$, (assuming $c=1$, which is not necessary, and is shown here to simplify the derivation) from which follows that

$u_n = u_{xx}$. Substituting this into the Taylor Series, we get the following expansion

$$u(x, t + \Delta t) = u(x, t) - \Delta t u_x(x, t) + \frac{(\Delta t)^2}{2!} u_{xx}(x, t) - \dots \quad (3)$$

This new equation provides a method of determining the position of the wave at later points in time in terms of space derivatives. Therefore, one of the primary focuses of the research involves approximating the space derivative u_x and u_{xx} . Finite differences [2,3] can be used to approximate derivatives using a finite number of data points. When using four points in space to approximate the u_x , there are four finite difference approximations that could be used to approximate u_x , each using different data sets (stencils). The four stencils that could be chosen are relative to the point that is to be approximated, say for example $u_x(x, t + \Delta t)$ for a given value of x (denoted by \bullet in the following description). The first possible stencil contains the point at x , and the three points to the left ($\circ\circ\circ\bullet$). The next stencil contains the point at x , two points to the left, and one to the right ($\circ\circ\bullet\circ$). The third stencil contains one point to the left, and two to the right of x ($\circ\bullet\circ\circ$). The fourth stencil contains the point at x and three points to the right ($\bullet\circ\circ\circ$). For one-way waves that travel to the right, this last stencil is not useful due to upwinding. Upwinding [1] requires that one always select a stencil at x that contains the point next to it in the direction in which the wave is coming. The method in which one selects from among the three remaining stencils may be used to determine derivative approximations, which is another primary interest in this research.

ENO Method

One method (developed in the mid 1990's, see [4]) is referred to as the Essentially Non-Oscillatory (ENO) method. Each derivative will be approximated using one of the stencils presented above. The ENO method will choose the stencil by building it up, starting with the point at x , (\bullet). ENO will always choose to add the point next to it in the direction in which the wave is coming due to upwinding (with $c > 0$, the stencil is now $\bullet\circ$). ENO then selects from two sets of three points. One set consists of the first two points plus the point to the left ($\circ\circ\bullet$), and the other set contains the first two points plus the point to the right ($\circ\bullet\circ$). If an interpolating parabola is generated for each of the two data sets, then ENO selects the set that forms the flatter (smaller in magnitude second derivative) parabola. At this point, only one more point needs be added to form the completed stencil. ENO adds either a point to the left of the selected three, or a point to the right, by selecting the data set that is interpolated by the cubic polynomial which has the smallest, in magnitude, third derivative. Note that two steps are needed to find four points to use.

Linear Deviation Method

As a result of this research, an alternative to the ENO method was found. This new method, the linear deviation method, selects the data with the least linear deviation and only uses one step to find four points to use (choosing one of the three upwind stencils outlined above). Once the upwind point is determined, this new method then uses the two points ($\circ\bullet$ for $c > 0$) to form a line. The two points to add are chosen by selecting the pair that has the least linear deviation (in the \mathcal{L}_1 sense) from the line. In other words, the set of two points that are closest to the line. The amount of deviation from the line and each point is calculated by subtracting the value of the line at that point and the value of the data at that point, and then applying the absolute value. Next, the deviations are summed for the three sets of two points. The set with the least amount of summed linear deviations is chosen.

The Linear Deviations method can also jump from two to five points using this same concept. When jumping from two points to four, this method is referred to as the 2-pt method, since 2 additional points are added. It follows that when jumping from two points to five, the method is referred to as the 3-pt method, since 3 additional points are added. While the 3-pt methods lead to similar results, only the 2-pt method will be discussed in detail.

Calculating the Space Derivative

With either method (ENO or the 2-pt Linear Deviation) the second derivative of u with respect to x (u_{xx}) can be approximated in two different ways. With each method (ENO or the 2-pt), one could first find u_x and then again apply the same method to the u_x data to arrive at an approximation to u_{xx} . A second approach would be to modify the routines so that the second derivative is approximated directly using linear algebra [4]. The first approach requires nearly twice as many calculations, but attempts to adapt to the changes in the derivative as well as the changes in the function (data). Acceptable increases in computational complexity are those increases which lead to a better approximation. On certain waves, calculating the first derivative twice to approximate u_{xx} leads to a better approximation, while on other waves, calculating the second derivative directly leads to a better approximation. Thus, examples of both types of waves are presented. Each wave will be approximated using ENO and 2-pt. ENO_a and 2-pt_a calculate the first derivative twice to get the second derivative. ENO_b and 2-pt_b use linear algebra to calculate the second derivative directly.

It should be noted that it was unexpected that calculating the second derivative directly using linear algebra would give better approximations on certain waves. The stencils are chosen for ENO and Linear Deviation in a non-linear fashion. Since it is not linear, calculating the second derivative directly does not give the same results as calculating the first derivative twice. Even though it may seem that taking the derivative twice would better adapt to the data, on some waves this was shown to not be the case.

Introduction to the Analysis

To determine which method gives better results, the various methods were tested against a set of problems for which the exact solution is known. The approximation and the exact solution were compared using the l_1 , l_2 and the l_∞ norms [2]. The l_1 norm sums the absolute value of the differences between the true solution and the approximation at each of the discrete spatial points. The l_2 norm calculates the sum of the squared differences between the true solution and the approximation (least squares), and the l_∞ norm finds the maximum difference between the true solution and the approximation (mini-max). Thus, error norms that are closer to zero imply better approximations.

The primary determining factors for method comparison are the error norms and the amount of work (number of calculations needed) to calculate u_x and u_{xx} (all other aspects of the algorithms for each method were the same).

ANALYSIS

Eight data sets were used to test the methods for which the exact wave was known for all steps in time. Each data set consisted of discrete data. In the calculations, a grid spacing of $\Delta x = 1/101$ and $\Delta t = 2\Delta x/c$ were used. For each data problem, periodic boundary conditions were assumed and the approximation was compared to the initial data after two revolutions (with wave speed $c=1$, the exact solution would exactly match the initial data). Using the true solution, and the approximation arrived at by using a certain method moved from time step to

time step, comparisons can be made using the norms to see how well the solution is approximated. For reference, the linear deviation method is compared to ENO as the standard method.

As mentioned before, two types of waves will be examined. The first is the “ramp” data set shown in Figure 1. This is a very simple data set that consists of four lines connected together. Where the lines are connected, the function is continuous, but not differentiable. This affects the convergence of the Taylor series expansion of the wave. Also, periodic boundary conditions are set so that when the wave reaches the right side of the box, it will wrap back into the left side of the box.

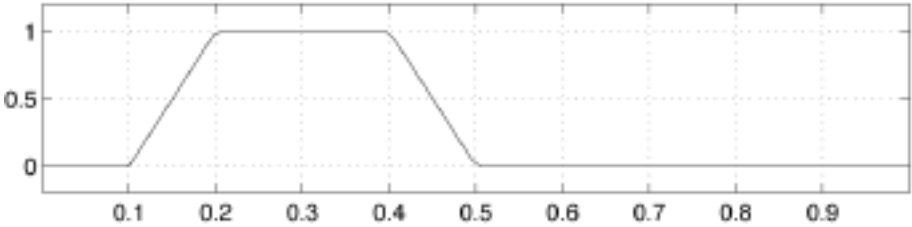


Figure1 :The ramp function with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

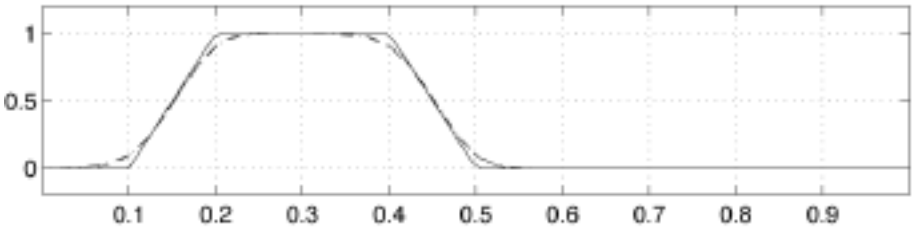


Figure 2:ENO_a and 2-pt_b after 505 steps in time (2 revolutions) with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

Figure 2 contains a comparison of the true solution (the solid line), ENO_a (the dotted dashed line) and 2-pt_b (the dashed line). It is hard to visually distinguish between the two methods and the true solutions except at the corners where the original function is not differentiable. Since a visual comparison is difficult, the norms are presented in the following table.

Norm	ENO _a	ENO _b	2-pt _a	2-pt _b
l_1	1.8668	1.527	1.111	1.2269
l_2	0.3398	0.2879	0.2642	0.2502
l_∞	0.1169	0.1008	0.0940	0.0910

From the above norms, it can be seen that the 2-pt_b method outperforms the other methods. This can also be seen better visually if the errors of the methods are plotted. To find the error, the approximation is subtracted from the true solution and the resulting function is plotted. Therefore, the closer the approximation, the closer to zero the curve in the graph will be.

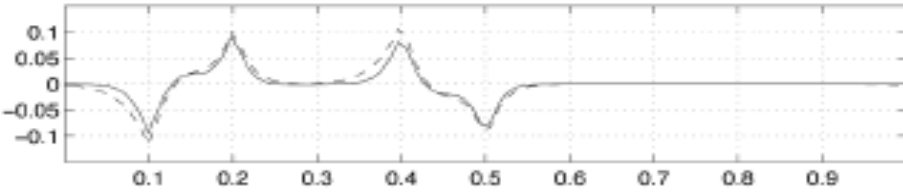


Figure 3: Error in ENO_a and $2-pt_a$ after 505 steps in time (2 revolutions) with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

Figure 3 contains the error plots for the methods that approximate the second derivative with respect to space by calculating the first derivative twice (the subscript a methods).

The error for ENO_a is the dashed line and the error for $2-pt_a$ is the solid line. From this graph, it can be seen that the $2-pt_a$ method yields a better approximation than ENO_a , a fact observed in the norm table

Figure 4 contains the error plots for the methods that approximate the second derivative with respect to space by using linear algebra to calculate it directly.

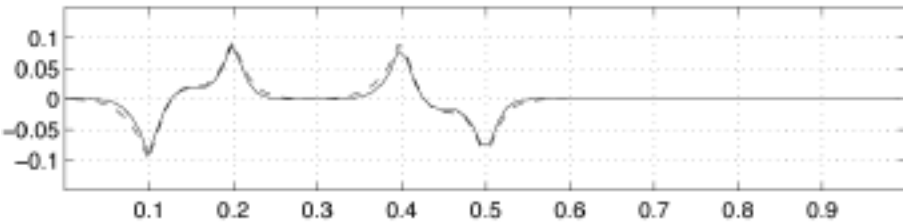


Figure 4: Error in ENO_b and $2-pt_b$ after 505 steps in time (2 revolutions) with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

The error for ENO_b is the dashed line and the error for $2-pt_b$ is the solid line. From the graph, it can be seen that the $2-pt_b$ method has a closer approximation than ENO_b . This is also shown in the norms.

It should be noted that the $2-pt_b$ method outperforms the other methods. This is desirable because compared to the ENO methods, linear deviation methods take less calculations since they “jump” from two points to four in just one step rather than two. This is also good because the $2-pt_b$ method takes fewer calculations than $2-pt_a$ due to calculating the second derivative directly in one step instead of two. So, all in all, the $2-pt_b$ method takes the least

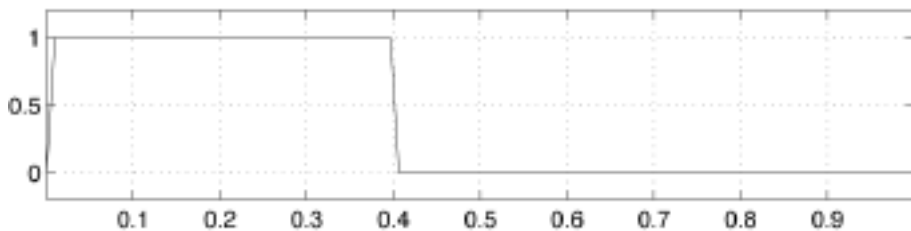


Figure 5: Step function with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

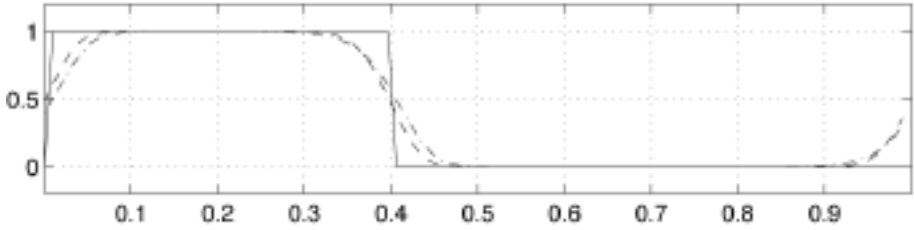


Figure 6: ENO₃ and 2-pt₃ after 505 steps in time (2 revolutions) with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

computations of the four methods and yields the best error norms.

As mentioned earlier, some waves are approximated better using approximations that calculate the second spatial derivative by taking the first derivative twice. One such wave is the step function shown in Figure 5.

The step function consists of 3 lines, which are not connected (there are vertical jumps in

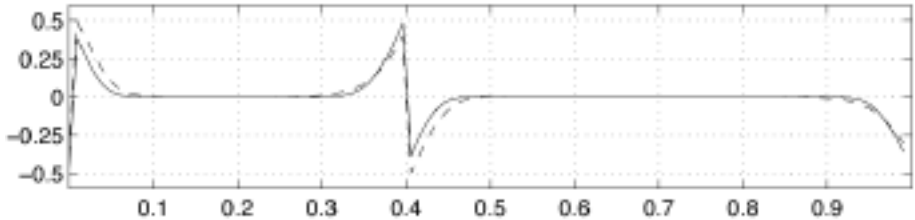


Figure 7: Error in ENO₃ and 2-pt₃ after 505 steps in time (2 revolutions) with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

the data). This function is neither continuous nor differentiable at the points where the jumps occur. This too affects the convergence of the Taylor Series expansion of the wave.

Figure 6 contains a comparison of the true solution (the solid line), ENO₃ (the dotted dashed line) and 2-pt₃ (the dashed line). With this graph, it can be seen which method does better. The norms associated with the various approximations are contained in the following table.

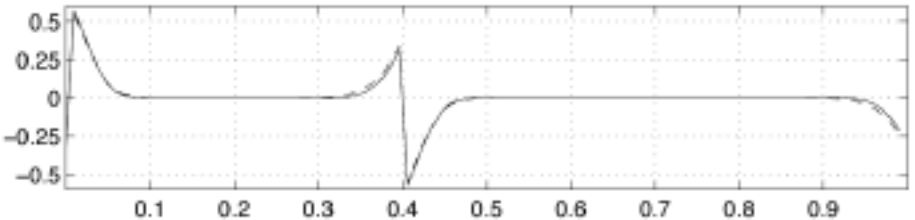


Figure 8: Error in ENO₃ and 2-pt₃ after 505 steps in time (2 revolutions) with $\Delta x = 1/101$ and $\Delta t = 2\Delta x/5$.

Norm	ENO _a	ENO _b	2-pt _a	2-pt _b
l_1	5.9266	5.2826	4.7778	4.8951
l_1	1.3114	1.2784	1.1953	1.2758
l_∞	0.4941	0.5458	0.4851	0.5703

Figure 7 contains the graphs of the errors involved with the methods that approximate the second derivative with respect to space by using linear algebra to calculate it directly (the subscript b methods).

The error for ENO_b is the dashed line and the error for 2-pt_b is the solid line. The norms show that the 2-pt_b method does better on the first two norms, but ENO_b does better on the infinity norm. Figure 8 contains the errors for the methods that approximate the second derivative with respect to space by calculating the first derivative twice.

The error for ENO_a is the dashed line and the error for 2-pt_a is the solid line. From this graph, it can be seen that the 2-pt_a method has a closer approximation than ENO_a.

Recall that since 2-pt_a jumps from using two points to four in only one step, it requires fewer calculations than ENO_a. Therefore, it can be said that 2-pt_a outperforms ENO_a.

CONCLUSIONS

The various methods described in this paper were applied to a number of different wave data sets. It was found that certain waves are better approximated using different methods. It should be noted, however, that all waves were better approximated using the linear deviation methods rather than the ENO methods. This is desirable since linear deviation methods are less complex and require fewer computations. Even though the linear deviation methods do not give highly significant gains in accuracy, it was possible to increase the accuracy (as compared to ENO) with fewer computations.

Some preliminary research has been completed using the 3-pt method that “jumps” from two points to five points by adding 3 additional points. By comparing the error norms produced, the linear deviation methods were again shown to outperform the ENO methods, again with fewer computations.

REFERENCES

1. Bleistein, *Numerical Methods for Wave Phenomena*, Academic Press, 1984.
2. Burden and Faires, *Numerical Analysis, Sixth Edition*, Brooks/Cole, 1997.
3. Cheney and Kincaid, *Numerical Mathematics and Computing*, Third Edition, Brooks/Cole, 1994.
4. Hoar, *An Adaptive Stencil Finite Difference Method for First Order Linear Hyperbolic Systems*, PhD Thesis, Montana State University, 1995.
5. Riley, Bruce *An Introduction to Computational Mathematics*, University of Wisconsin-La Crosse, Spring 1998.