The Impact of Teaching Algebra with a Focus on Procedural Understanding

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ABSTRACT

Present research indicates that teaching math for understanding may yield positive gains in student learning and retention of knowledge. In the 2007-2008 school year, nine classes used a student-centered framework for teaching math with an emphasis on understanding. Five comparison classes were selected as a baseline. Students in each of these 14 classes took three algebra tests throughout the year. We developed task-specific rubrics for scoring all three tests, blinded ourselves to the treatment condition by assigning random codes to all tests, and proceeded to score the 830 tests. Statistical analysis revealed that over the course of one-year study period the treatment students made significantly greater gains than comparison students on the skill questions; there were no differences in gains on the understanding questions. Many of the students, regardless of treatment, demonstrated deficiencies in basic algebraic proficiencies, including predicting, interpreting, and evaluating the correctness of their answers. These results warrant further investigation into the effectiveness of teaching for understanding.

INTRODUCTION

Mathematics is a multidimensional discipline. It is about procedures, concepts, and number facts. It is about predicting, estimating, and verifying. It can be algorithmic, and yet it is also robust and permits flexibility and seeks efficiency. It is grounded in reality and yet it defines a reality of its own. Too often, school mathematics overlooks these intriguing dimensions and focuses on training students to become proficient only at following prescribed recipes in the context in which they were learned. Yet such a limited view of proficiency produces students who have difficulty explaining how they know their work is legitimate, what alternate methods could have been used, what approach would be the most efficient, or how to communicate their answer and verify that it is reasonable. Efforts aimed at teaching for procedural understanding seek to redirect students to delve deeper into the subject matter where they can begin to develop a multidimensional understanding of the mathematical procedures they study. This deeper knowledge of procedures has been characterized over the years as connected knowledge, metacognition, mathematical literacy, and as algebraic habits of mind (Brozo & Simpson, 1999; Burke et al., 2001; Cuoco et al., 1996; Goldenberg, 1996; Hiebert & LeFevre, 1986; Schoenfeld, 1992). We have adopted the term that has become popular in recent literature on mathematics education: *procedural understanding* (Baroody et al., 2007; Hasenbank, 2006; Star, 2005, 2007). By any name, this deeper 'procedural understanding' is directly contrasted with shallow, disconnected, and simply memorized knowledge. Research from both cognitive psychology and mathematics education suggests that such shallow knowledge is short-lived and does not transfer well to contexts different from the learning environment (Bower, 2000; Greeno, 1978; Hiebert & Carpenter, 1992), while knowledge that is understood lasts longer and can be used flexibly (Star & Siefert, 2002).

In 2001, the National Council of Teachers of Mathematics (NCTM) published its Navigating Through Algebra in Grades 9-12. In its pages was a concise framework for what the algebraically literate student would know and be able to do. Dr. Hasenbank and his colleagues modified the NCTM framework to make it more student centered for use in a teaching experiment in college algebra (Hasenbank, 2006); the resulting student-centered framework for teaching mathematics for understanding consisted of the following eight questions:

1a) What is the goal of the procedure?
1b) What sort of answer should I expect?
2a) How do I carry out the procedure?
2b) What other procedures could I use?
3) Why does the procedure work?

4) How can I verify my answer?
5) When is this the “best” procedure to use?
6) What else can I use this procedure to do?

This *procedural understanding framework* helps illustrate what students should understand about algebra procedures by the end of their coursework**,** and it provides specific questions for use in both instruction and assessment.

Teaching for understanding in the mathematics classroom involves building up such understandings of certain mathematical procedures. Examples of procedures include factoring by grouping, graphing in slope intercept form, solving an algebraic equation for a variable, and so on. For instance, if a student was asked to solve $3x+4 = 7$, they would be expected to find an answer of $x = 1$. A series of possible understanding follow-up questions include:

- “Discuss as many ways as possible for checking to see if your solution is correct,” (Possible answers: Graph the lines $y = 3x+4$ and $y = 7$ and confirm they intersect when $x = 1$; Evaluate the left hand side of the equation at $x = 1$ and confirm it is equal to 7).
- “Can you solve for x in a different way?” (Possible alternate methods: educated guess and check, graphing, or making a table of values)
- “How could you use estimation to decide whether the answer should be positive or negative?” (Possible answer: It should be positive because we need to add to the 4 to get up to 7)
- “How might you convince a skeptic that the steps you used are legitimate?” (Possible answers: We solve by applying the additive and multiplicative properties of equality, or we could make and use a physical model to demonstrate the steps).

The research so far is fairly limited in measuring the short- and long-term impact of teaching for understanding. One reason for this limitation has been a historical linking of the descriptor “understanding” with knowledge that is conceptual (but not procedural) in nature (Star, 2005). It is becoming clear that *both* procedures and concepts can be understood along a continuum ranging from shallow to deep; indeed, Hasenbank’s adaptation of the NCTM framework defines a series of questions that can be used to assess the depth of a student’s procedural understanding.

The recharacterization of procedural knowledge (Hiebert & Handa, 2004; Star, 2005) has begun to influence new curricula and has informed the debate on what effective teaching can and should look like. Yet teaching for understanding can be challenging to accomplish in a classroom for a number of reasons: the time & training needed for developing materials, student resistance to new ways of teaching, and the development of new models for assessing for understanding. But despite the reasons against explicitly teaching for understanding, there is a growing body of evidence that suggests explicitly teaching for understanding has benefits for students. According to Hiebert & Carpenter (1992), if existing knowledge is understood, new knowledge is easier to learn and understand. Students can learn new content more quickly once previous procedures and concepts are understood. In addition to this, knowledge that is understood lasts longer and can be applied in a variety of situations (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Van Hiele, 1986). With understanding, students are gaining a more diverse education that can serve them in a variety of ways. Furthermore, understanding and skill support one another, and these ties lead to more robust, longer lasting knowledge (Rittle-Johnson, *et al.*, 2001). Lastly, research shows that with understanding procedures are executed ‘intelligently’ and with fewer errors (Star & Siefert, 2002). The existing literature on teaching for understanding supports it as a worthwhile pursuit in mathematics education.

METHOD

In the summer of 2007, 13 teachers participated in a summer institute on how to teach for understanding in their classrooms. During this course, the teachers learned the theory behind teaching for understanding, they learned questioning strategies for engaging students in key algebraic habits of mind as defined by Hasenbank’s modified NCTM framework, and they learned to assess understanding to reinforce its value and measure its growth. Follow-up training days during the academic year focused on the classroom implementation of the strategies learned during the summer institute.

After the summer institute, participating teachers each selected one or two focus classes they would teach using the framework for understanding. Five comparison teachers were also recruited to act as a control group. The students in each focus class took three algebra tests throughout their academic year to assess their mathematical skill and understanding. The tests each consisted of approximately 17 questions and a mix of both skill questions and understanding questions. The treatment student sample consisted of five algebra classes, two geometry classes, one physical science class, and one AP calculus class. The comparison sample consisted of four algebra classes and one geometry class.

Data analysis began with creating task-specific rubrics for each question on the three tests. The rubrics were all based on the following 0 to 3 scale:

0: No degree of understanding
1: Low degree of understanding
2: Moderate degree of understanding
3: High degree of understanding

We then reviewed a subset of the 830 tests and began to create the task-specific rubrics. In order to develop a reliable rubric for scoring the diverse set of responses, we generated a series of preliminary rubrics and revised them based on a pilot scoring of student responses. The final rounds of pilot grading served as a reliability check between graders (discussed in further detail below). Eventually we agreed upon a set of task-specific rubrics that we felt would allow us to consistently code the wide variety of answers in a way that would accurately assess students' understanding.

To illustrate the assessment format and the typical rubric design, we have included in Figure 1 a sample skill question and subsequent understanding question, together with their associated rubrics:

<u>Skill Question:</u>

Evaluate (2/3)x when x = (9/4) . Simplify your answer if possible.
Correct Solution is 3/2, or 1½, or 1.5.
(Note: 18/12 is incorrect because it has not been simplified.)

<u>Rubric:</u>

Possible errors:

- Finding a common denominator
- Computational error
- Simplifying incorrectly

0: Blank or no honest effort (e.g. "I don't know (IDK.)")
1: Two or more errors, or used cross multiplication to solve, or an incomplete attempt, or a solution that is way off track.
2: Only one error is present as listed above.
3: Correct answer is given in lowest terms.

<u>Understanding Question:</u>

Explain how you could use estimation to see if your previous answer was wrong.

<u>Rubric:</u>

Sample responses:

- I can use decimals to estimate the problem as ".7 x 2 = 1.4".
- I can estimate the problem as 2/3 of 2¼, so it should be between 1 and 2.
- The problem is a fraction (less than one) times a number a little greater than 2, so it should be less than 2.
- I can estimate the problem as "a little less than 1" times "a little more than 2", so the answer should be close to 1 x 2 = 2.

0: Blank or no honest effort (e.g. "I don't know (IDK.)")
1: Response is vague, confusing, contains little insight (e.g. "Use my calculator"), OR uses estimation after performing the multiplication (e.g. I got "1 and 6/12", and 6 is about half of 12, so 1½ is right).
2: Response uses the relative magnitudes of the fractions (e.g., use decimal approximations, or rounds 9/4 to 2) but <u>omits details</u> or <u>contains minor mistakes</u> or oversights that detract from the overall argument.
3: Response is similar to one of the sample responses, or uses other techniques that produce a reasonable estimate.

Figure 1. Sample Rubrics

Before we scored the tests, we blinded ourselves to the treatment condition to ensure we were not unintentionally entering bias into our coding. We stripped identifying information from the tests and assigned a random code to each student. Because we needed to divide up the workload between three graders, we also needed to make sure we were scoring items consistently. To test this we conducted a pilot grading of all tests from students whose four-digit random code began with 0 (that is, approximately 1/10 of the tests). We found Cohen's *kappa* for inter-rater agreement on the three tests to be 0.867, 0.843, 0.881, respectively. A *kappa* statistic above 0.8 is considered ‘Almost Perfect’ agreement (Landis & Koch, 1977). Since this was satisfactory, we continued to score the remaining tests.

To better illustrate the grading process, we have included two student responses to the sample questions we included in Figure 1. These are presented in Figure 2 and 3, below. The assigned score for each response is also shown.

Skill Question:

Evaluate (2/3)x when x = (9/4) . Simplify your answer if possible.
Correct Solution is 3/2, or 1½, or 1.5.
(Note: 18/12 is incorrect because it has not been simplified.)

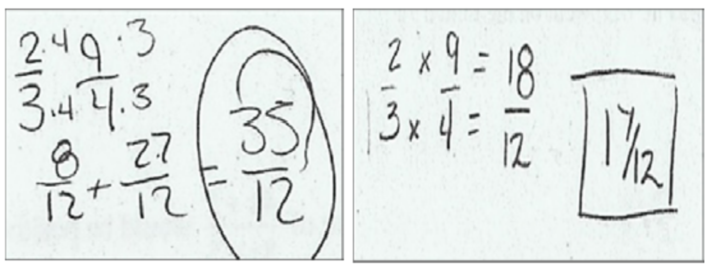

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Figure 2. Sample Skill Question with Student Responses

Understanding Question:

Explain how you could use estimation to see if your previous answer was wrong.

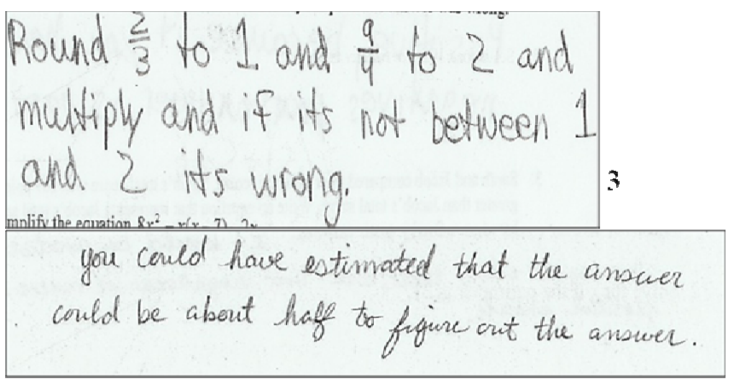

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Figure 3. Sample Understanding Question with Student Responses

RESULTS AND DISCUSSION

Our analysis consisted of comparing gain scores from test 1 to test 3 for both the skill and understanding subscores. The data presented here represent only the algebra classes to permit comparison between treatment and control groups. The skill questions in algebra illustrate the attributes that are often associated with the subject, and these are certainly an important component of algebraic fluency. A repeated measures ANOVA indicated significant differences between groups. Follow-up t-tests revealed a significantly greater gain in skill from test one to test three ($p = .003$) in favor of the treatment group. The treatment group started with an average skill score below the control group but finished with a higher average skill score (see Figure 4).

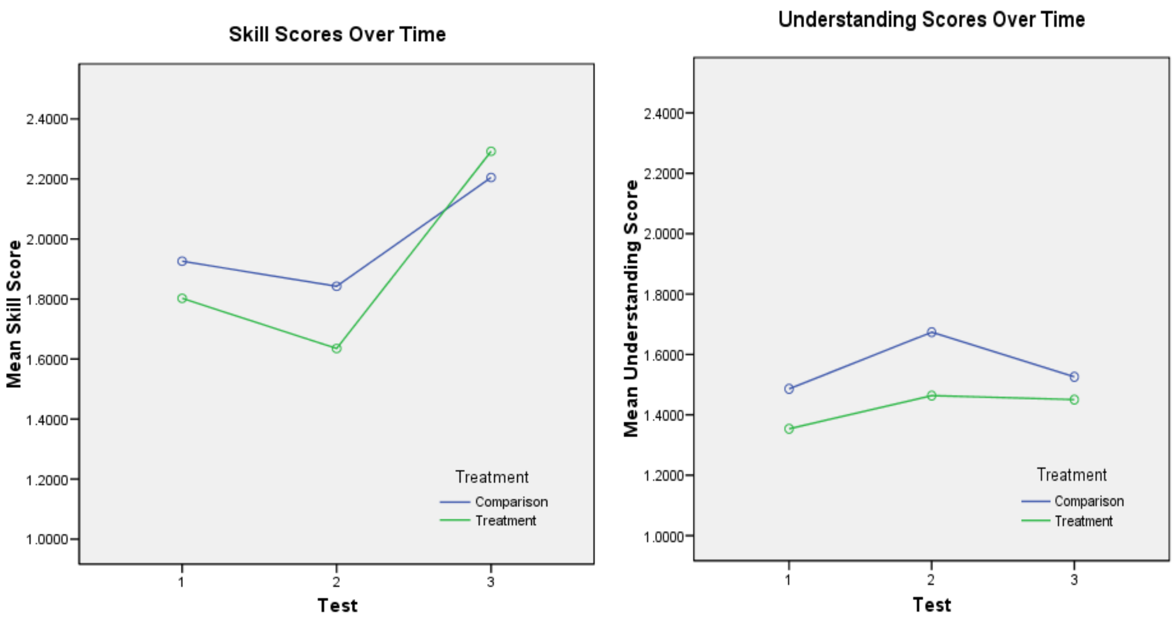

Figure 4. Skill and Understanding Subscore Results

We next looked at the understanding results for the students in algebra. The t-test of the gain score in understanding from test 1 to test 3 revealed no significant difference in gain scores between groups ($p = .413$). This result shows that the treatment group did not make improvements in their understanding scores over the comparison group. Figure 5 displays this result above. It is worth noting that, as the figures suggest, the overall understanding scores were statistically well below the overall skill scores ($p < .001$), indicating that regardless of treatment condition the students scored significantly lower on our understanding questions than they did on our skill questions. Apparently additional work is needed to develop methods that raise students' ability to answer deep questions about the procedures they are learning.

The fact that we observed greater gains in skill for the treatment group but no differences on the understanding questions led us to wonder what the relationship between skill and understanding questions was for our students, particularly since many of the understanding questions produced very poor results. To explore this discrepancy between skill and understanding, we produced crosstabs of skill scores vs. understanding scores for our students, regardless of treatment condition. The skill and understanding questions included in the crosstab are shown for reference in Figure 5 (below); these results were taken from the year-end test.

Skill question: If $3x + 4y = 14$, and $y = -(5/2)x$, what is the value of x?

Answer: $x = -2$

Understanding question: What does the answer to the previous problem tell you about the graphs of the two equations?

Answer: The graphs of the two lines will intersect at the point (-2,5).

Figure 5. Skill and Understanding Questions used on Crosstabs

Table 1 (below) shows a crosstab of skill score vs. understanding score for all students (including algebra, geometry, physical science, and AP calculus) who took the three tests.

Table 1. Understanding vs. Skill Crosstab Results

		Understanding				
		0	**1**	**2**	**3**	**Total**
Skill	**0**	40 (15.6%)	5 (1.9%)	0 (0.0%)	0 (0.0%)	45 (17.5%)
	1	35 (13.6%)	73 (28.4%)	11 (4.3%)	1 (0.4%)	120 (46.7%)
	2	5 (1.9%)	36 (14.0%)	3 (1.2%)	0 (0.0%)	44 (17.1%)
	3	8 (3.1%)	32 (12.5%)	5 (1.9%)	3 (1.2%)	**48 (18.7%)**
	Total	88 (34.2%)	146 (56.8%)	19 (7.4%)	**4 (1.6%)**	257 (100%)

The highlighted cells in the crosstab reveal that, even at the end of year, most students demonstrated a low level of understanding on this algebra task. The crosstab shows that only 4 students could correctly answer the understanding question, while 48 students could correctly answer the related skill question. Other crosstab results revealed similar results. We interpret this as evidence that many students had a very limited understanding of the processes of algebra, especially in their ability to predict, estimate, and make sense of their answers. Moreover, students had difficulty connecting the problems they solve to the underlying meaning or alternate representations. Such limited understanding is characteristic of the fragile learning we noted in our introduction, and it suggests a greater emphasis is needed on procedural understanding in the K-12 curriculum.

In addition to this, the total understanding average score from all algebra students on test three was only 1.44. This means that after a full year of algebra, most students were at either a low or moderate degree of understanding. Several understanding questions in test 3 produced average scores around 0.7, indicating a very low degree of understanding (it also indicates that many students left these questions blank). Our research shows that certain algebra procedures are simply not being understood, which further supports the need for more effective student-centered implementations of the principles of teaching for understanding.

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