Mathematics of Music

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Music is the pleasure the human soul experiences from counting without being aware that it is counting.
    -Gottfried Wilhelm von Leibniz, a German mathematician who co-discovered calculus.

ABSTRACT: A history of mathematics includes early connections with music and the basic physics of sound. Mathematics is present in the natural occurrence of the ratios and intervals found in music and modern tuning systems. In this paper we will examine both the mathematics and music background for these ideas. We will examine the Fourier Series representations of sound waves and see how they relate to harmonics and tonal color of instruments. Some modern applications of the analysis will also be introduced.

1. INTRODUCTION

To those who have studied mathematics or music in any depth, it has certainly been mentioned that mathematics and music are deeply connected at the roots. This often seems to be taken as an unexplained given. If asked, however, to relate some specifics of the connection, most students of either principal may not have much to say. This paper is aimed at exploring the relation between mathematics and music, including a specific discipline of mathematics, Fourier Analysis. Fourier Analysis can be used to identify naturally occurring harmonics (which are, simply put, the basis of all musical composition), to model sound, and to break up sound into the pieces that define it.

2. HISTORY OF FOURIER ANALYSIS AND MUSIC

The connection between mathematics and music goes back at least as far as the sixth century B.C. with a Greek philosopher named Pythagoras. Most people will know him best for the Pythagorean Theorem in relation to geometry or trigonometry, but this is not his only claim to fame. He studied music as well, and understood the arithmetical relationships between pitches. It is said that he discovered the relationship between number and sound. He believed that numbers were the ruling principal of the universe. As the human ear is unable to numerically analyze sound, Pythagoras turned to the vibrating string, explored the ideas of the length of strings and pitches, and found simple ratios relating harmonizing tones [Forster]. A musical tuning system is based on his discoveries, and will be discussed below.

These ratios and harmonizing tones come from the harmonic series, which will be discussed in detail later. The basic idea for now, is that harmonics are tones that have frequencies that are integer multiples of the original tone, the fundamental tone. The fundamental and its harmonics naturally sound good together. Each tone has a harmonic series, which can be used to fill in a scale of notes; western music is based on harmonics. When a note is played on an instrument, due to the physics of the sound waves, we don't hear only that tone; we hear the played tone as the fundamental, as well as a combination of its harmonics sounding at the same time.

After Pythagoras discovered harmonics, many more explored the idea more thoroughly. At least two unassociated men took significant steps in defining harmonics. The first of these is Marin Mersenne (1588-1648), a French theologian, philosopher, mathematician, and music theorist. Some sources say he discovered harmonics, which he called *sons extraordinaire*, but in actuality, he defined the harmonics that Pythagoras had already found.

He mathematically defined the first six harmonics as ratios of the fundamental frequency, 1/1, 2/1, 3/1, 4/1, 5/1, and 6/1, or the first six integer multiples of the original tone's frequency [Forster]. Mersenne worked out tuning systems (this idea will be discussed below) based on harmonics. Also attributed with defining harmonics is Jean-Philippe Rameau (1683-1764), a French composer and music theorist. Rameau understood harmonics in relation to consonances and dissonances (intervals that sound good or clash) and harmonies. His paper, *Treatise on Harmony*, published in the 1720's, was a theory of harmony based on the fact that he heard many harmonics sounding simultaneously when each note was played. Whether these two men's work was related or not, Rameau's *Treatise on Harmony* created a stir that initiated a revolution in music theory. Musicians began to notice other harmonics sounding in addition to the played, fundamental tone, notably the 12th and 17th, which are the 12th and 17th steps in the scale of a given note, respectively [Sawyer].

In the 18th century, calculus became a tool, and was used in discussions on vibrating strings. Brook Taylor, who discovered the Taylor Series, found a differential equation representing the vibrations of a string based on initial conditions, and found a sine curve as a solution to this equation [Archibald].

Daniel Bernoulli (1700-1782) and Leonhard Euler (1707-1783), Swiss mathematicians, and Jean-Baptiste D'Alembert (1717-1783), a French mathematician, physicist, philosopher, and music theorist, were all prominent in the ensuing mathematical music debate. In 1751, Bernoulli's memoir of 1741-1743 took Rameau's findings into account, and in 1752, D'Alembert published *Elements of theoretical and practical music according to the principals of Monsieur Rameau, clarified, developed, and simplified*. D'Alembert was also led to a differential equation from Taylor's problem of the vibrating string,

$$\frac{\partial^2 y}{\partial x^2} = a^2 \frac{\partial^2 y}{\partial t^2} \qquad\qquad \textbf{(1)}$$

where the origin of the coordinates is at the end of the string, the x-axis is the direction of the string, y is the displacement at time *t* [Archibald]. This equation is basically the wave equation, which will be discussed later.

Euler questioned the generality of this equation; while D'Alembert assumed one equation to represent the string, Euler said it could lie along any arbitrary curve initially, and therefore require multiple different expressions to model the curve. The idea was that a simply plucked string at starting position represents two lines, which cannot be represented in one equation.

Figure a: The shape of a simply plucked string based on Euler's ideas.
http://www.marco-learningsystems.com/pages/sawyer/music.html

Bernoulli disagreed. After following Rameau's hint, he made an arbitrary mix of harmonics to get $y = a_1 sinxcost + a_2 sin2xcos2t + a_3 sin3xcos3t + \cdots$ [Sawyer]. His theory was that this equation could represent every possible vibration that could be made by a stretched string released from some position. Setting *t* = 0 should give the initial position of the string. Bernoulli said his solution was general, and therefore should include the solutions of Euler and D'Alembert. This led to the problem of expanding arbitrary functions with trigonometric series. This idea was received with much skepticism, and no mathematician would admit its possibility until it was thoroughly demonstrated by Fourier [Archibald].

This leads us to our celebrity, Jean Baptiste Fourier, Baron de Fourier (1768-1830), a French mathematician. Fourier, the ninth child of a tailor, originally intended to become a priest, but decided to study mathematics instead. He studied at the military school in Auxerre, and was a staff member at the École Normale, and then the École Polytechnique in Paris, and through a recommendation to the Bishop of Auxerre, he was educated by the Benvenistes, a wealthy, scholarly family. He succeeded Lagrange at the École Polytechnique and travelled to Egypt in 1798 with Napoleon, who made him governor of Lower Egypt. He returned to France in 1801 and published his paper *On the Propagation of Heat in Solid Bodies* in 1807 [Marks]. His theory about the solution to a heat wave equation, stating the wave equation could be solved with a sum of trigonometric functions, was criticized by scientists for fifteen years [Jordan]. What he came up with, effectively, was the Fourier Series. In 1812, the memoir of his results was crowned by the French Academy [Archibald].
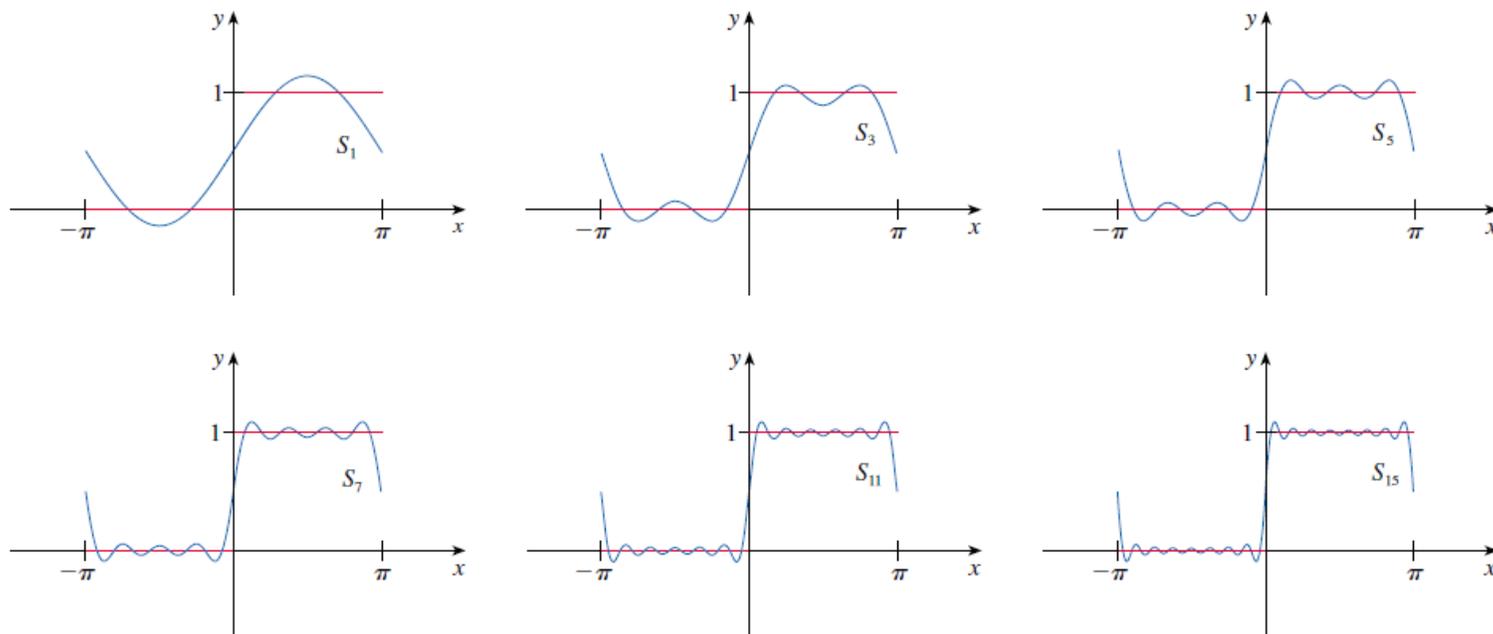
3. THE FOURIER SERIES

The Fourier Series is the key to the idea of the decomposition of a signal into sinusoidal components, and the utility of its descriptive power is impressive, second only to the differential equation in the modeling of physical phenomena, according to Robert Marks, author of *The Handbook of Fourier Analysis and Its Applications*. The basic Fourier Series takes the form of

$$f(x) \approx \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos \frac{n\pi x}{a} + B_n \sin \frac{n\pi x}{a} \right) \tag{2}$$

for a 2a-periodic function f(x). The coefficients are as follows:

$$A_0 = \frac{1}{a} \int_{-\pi}^{\pi} f(x)dx,$$

$$A_n = \frac{1}{a} \int_{-\pi}^{\pi} f(x)\cos \left(\frac{n\pi x}{a}\right)dx, \tag{3}$$

$$B_n = \frac{1}{a} \int_{-\pi}^{\pi} f(x)\sin \left(\frac{n\pi x}{a}\right) dx.$$

The idea is that as $n \to \infty$, the Fourier Series for f(x) will have enough terms that it will converge to the function. See the example below.

Figure b. $f(x) = \begin{cases} 1 \; when \; 0 < x < \pi \\ 0 \; when -\pi < x < 0 \end{cases}$

http://www.stewartcalculus.com/data/CALCULUS%20Early%20Transcendentals/upfiles/FourierSeries5ET.pdf

Figure b shows a simple piecewise equation in red, and the partial sums in blue (summed to a given n) of the Fourier Series of the function for *n*=1, 3, 5, 7, 11, 15. As n grows, the Fourier Series gives a closer approximation of the actual function.

The Fourier Series is the sum of trigonometric functions with coefficients specific to the function modeled. It is a sum of continuous functions, which can converge pointwise to a discontinuous function, as seen above, where each partial sum will be a continuous function. It can be used to solve and model complicated functions, and is a solution to the wave equation, which is a differential equation. The series can model any periodic function, but can also be used with other functions. The concept of sums of trigonometric functions to model other functions was not a new one in Fourier's time: Bernhard Riemann did some work with trigonometric functions to model other functions, as did Bernoulli.
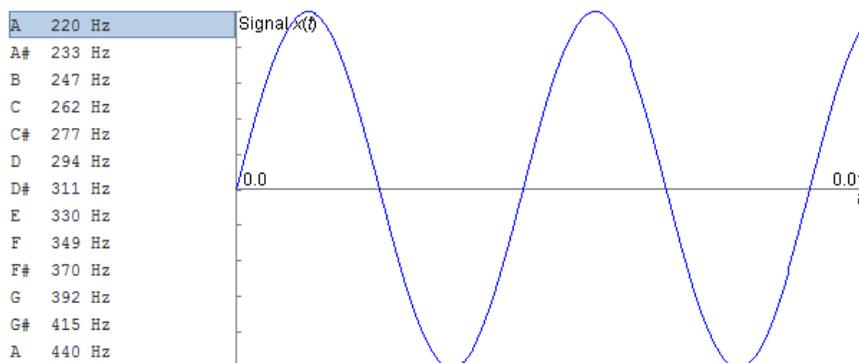
4. SOUND BASICS

The Fourier Series has many applications in the physical world, including that of modeling sound. Pure tones have frequency and amplitude, which determine the pitch and the strength of the sound, respectively. These are waves, and can therefore be represented by sinusoidal equations. Sounds are made of pure tones, combined in linear combinations to create more complex sounds, such as chords.

The vibrating strings and air columns on instruments obey the wave equation. The wave equation, as found by D'Alembert in equation 1, is a differential equation that examines the behavior of a piece of a string based on initial conditions, displacement, and release from rest.

As hypothesized by Bernoulli, the Fourier Series is a solution to the wave equation. This means that Fourier Series can be used to model the sound waves produced by vibrating strings and air columns.

Now that we've established the basics of sound, let's return to pure tones. Pure tones are, as implied by the name, pure and simple sound waves, which can be modeled by a single trigonometric function. For example, the pure tone of frequency 220 Hz, which would be an A in musical terms, has the following graph:
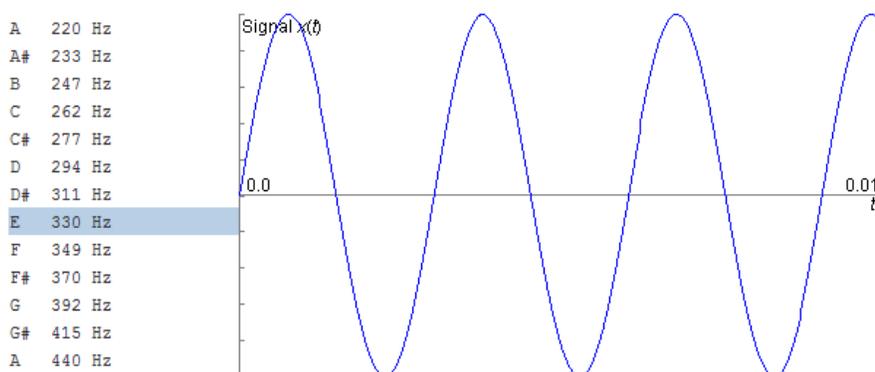

Figure c. $y = a \sin(2\pi (220)x)$, a is the amplitude of the wave.
The frequency of 220Hz would be an A in music.
http://www.jhu.edu/signals/listen-new/listen-newindex.htm

Compare this to a graph of an E at 330 Hz.

Figure d. $y = a \sin(2\pi (330)x)$
The frequency of 330Hz would be an E in music.
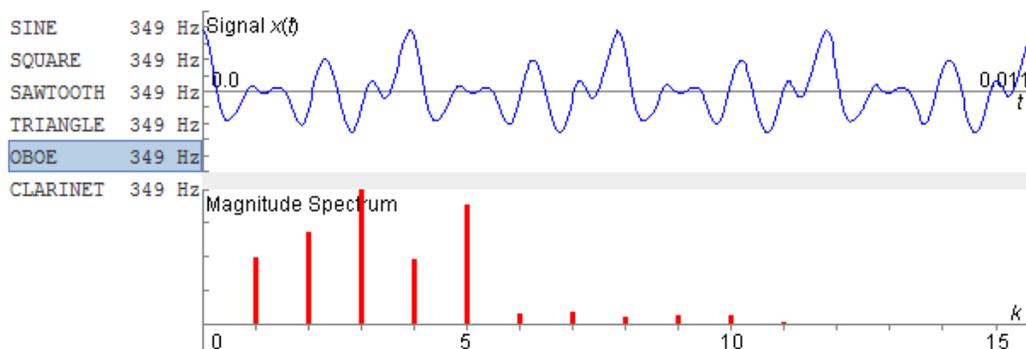http://www.jhu.edu/signals/listen-new/listen-newindex.htm

It can be seen from the graphs and the equations that the two notes differ by frequency, and therefore have different periods, but are both simple sinusoidal functions. If we listen to these tones produced by a computer, they sound very simple, and almost empty. We can't say which instrument would make these sounds, because an instrument cannot produce these pure tones, which we will discuss below. More complex tones, which sound less empty, are made by the addition of pure tones.
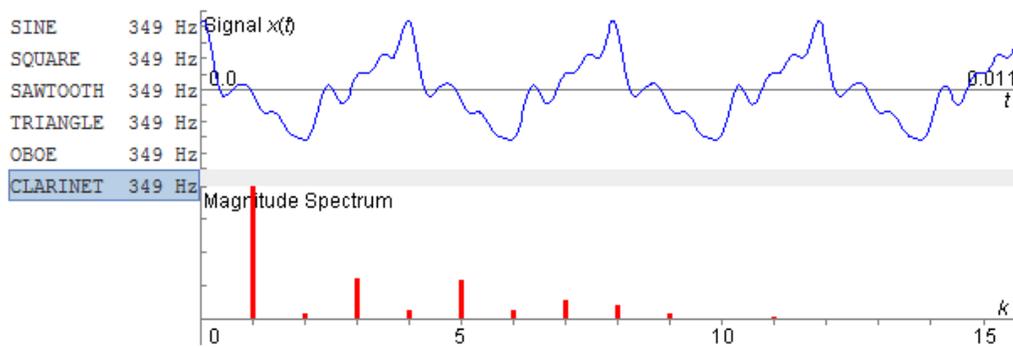
5. HARMONICS AND THE HARMONIC SERIES

This leads us to the discussion of harmonics. As mentioned above, when an instrument plays a note, the wave produced is not just a pure tone; it is a complex tone based on the physics of the instrument. When the note is played, the fundamental frequency is heard, as well as overtones, or harmonics. This is what determines the timbre of the instrument, or the tonal color; timbre is why different instruments playing the same note do not all sound the same. The instrument's timbre is what distinguishes its sound from that of a different instrument. The strength, or amplitude, of each harmonic is the difference we're hearing, since each note played includes the fundamental tone and some harmonics. In the graphs below, we see the harmonics and sound waves from the same note on different instruments. The blue wave is the sound wave, and the red bars are the amplitudes of respective harmonics, depending on k, the kth harmonic.

Figure e. This is a graph of a pure tone at a frequency of 349 Hz. Note it is only a sine wave, and there are no harmonics sounding, just the fundamental tone. This does not sound like a note played on an instrument because it is purely the fundamental tone, with no harmonics to add tonal color.
http://www.jhu.edu/signals/listen-new/listen-newindex.htm

Figure f. This is a graph of an oboe playing the same tone. Note the differences in the wave. This is due to the harmonics that are also heard, as seen below the graph.
http://www.jhu.edu/signals/listen-new/listen-newindex.htm

Figure g. This is a graph of a clarinet playing the same tone. The wave is similar to that of the oboe, but still notably different due to the different harmonics sounding with the fundamental. http://www.jhu.edu/signals/listen-new/listen-newindex.htm
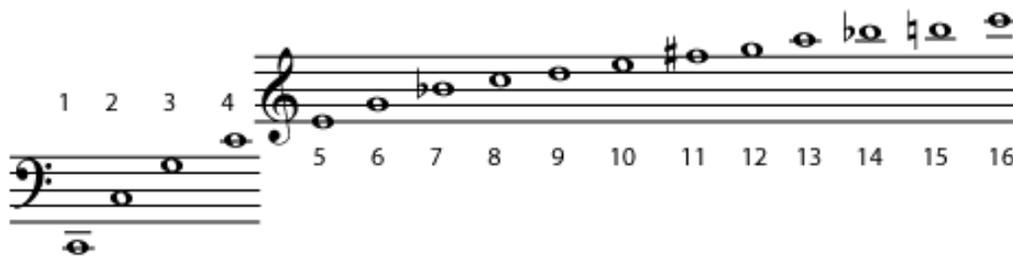
Harmonics are integer multiples of the fundamental frequency, and are therefore from the harmonic series of that frequency, which is the series of harmonics of the given fundamental. The frequency of the Mth harmonic of a tone is $(M + 1)u_0$, where u_0 is the fundamental frequency, which is defined as the lowest frequency allowed by the length of the air column or string. So to recap, the fundamental frequency is the pitch, or note, heard when the tone is played. The harmonics determine the timbre, or tonal color, of the sound. This is what differentiates the sounds of different instruments playing the same note.

The tonal quality, or timbre, of the sound includes its richness or harshness. This is not necessarily a technical definition, but rather how the listener would describe the sound. These two qualities can be directly attributed to the upper or lower harmonics. Sounds that contain more upper harmonics tend to sound brighter, or sometimes harsh; sounds that contain lower harmonics sound richer or softer.

The contributions of upper and lower harmonics can also be seen in the examination of white noise. White noise occurs when many or all equal-amplitude frequencies are sounded at the same time. If one removes the lower-register frequencies, the sound suddenly becomes much harsher, and it seems somewhat more bearable if the higher-register frequencies are removed. To listen to all of the above sound graphs, as well as white noise, visit http://www.jhu.edu/signals/listen-new/listen-newindex.htm.

So we've determined the importance of harmonics in sound, but we haven't yet discussed their origins in the harmonic series. The harmonic series is the series of tones created by multiplying a fundamental frequency by integers. This can be done based on any fundamental frequency, and each will result in a unique harmonic series. This is where pure intervals, intervals with frequencies related by small integer ratios, as found by Pythagoras when he cut strings in half, come from and where we get the ratios for them. The numerator of the ratios is the multiple of the fundamental frequency, and the denominator is the number of octaves between the two; we divide by this to put the tone in the same octave as the fundamental.
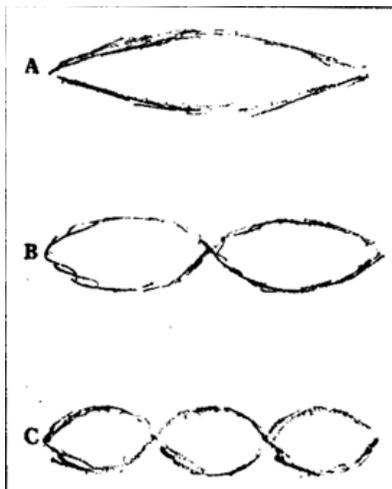
As an example, let's look at the harmonic series on C, where C is the fundamental note or frequency. The first harmonic plays a C an octave higher, which means that the frequency ratio between octaves is 2:1. The second harmonic is a fifth higher than that, with frequency three times the fundamental frequency. So by dividing by 2, we put the fifth in the same octave as the fundamental, since it is originally an octave too high, and thus to go up a perfect fifth, the ratio is 3/2. Similarly, the ratio for a perfect fourth in the same octave is 4:3. Going up the harmonic series will produce the notes of a major scale, where the first five tones are those of the C major triad, which is C, E, and G. A triad is a very strong musical feature in modern Western Music. So multiplying the fundamental by *n*, we go up the harmonics series. To go down the series, called the subharmonic series, we multiply by *1/n*. If we continue the harmonic series up and down, we will have the major and minor scales of notes, so we can fill in all of the notes of the chromatic scale, which would be like playing every key on a piano. All notes except for one, called the tritone or diablus en musica, devil in music. The tritone is the one and only note between a perfect fourth and a perfect fifth, and by our Western standards, it sounds awful. This is F# for the C harmonic series. In order to finish the chromatic scale, we compile the major and minor scales from the harmonic and subharmonic series into one octave, and we fill in the one missing note.

6

Figure h. This is the first 16 pitches of the harmonic series on C. The number below each note is the integer multiplied by the fundamental frequency (the frequency of C, here) to obtain the note's frequency. The harmonic series is unique to the fundamental note, but the same set of intervals and frequency ratios will be found in every harmonic series.
http://cnx.org/content/m11639/latest/

When Pythagoras was investigating strings, these small integer ratios (above) are the ones he found. To the left is an illustration of the ratios on a string. The first image is the full length of the string vibrating. The second is the string divided in two, which doubles the frequency and produces a tone an octave higher. The third is the string divided in three, which triples the frequency, and produces a tone another fifth higher. As these harmonics are naturally occurring based on the physics of sound, they sound naturally pleasing, particularly in the lower harmonics. The harmonic series is not just a convenient idea created by music theorists; it actually exists naturally, in the physics of sound, as Pythagoras discovered.
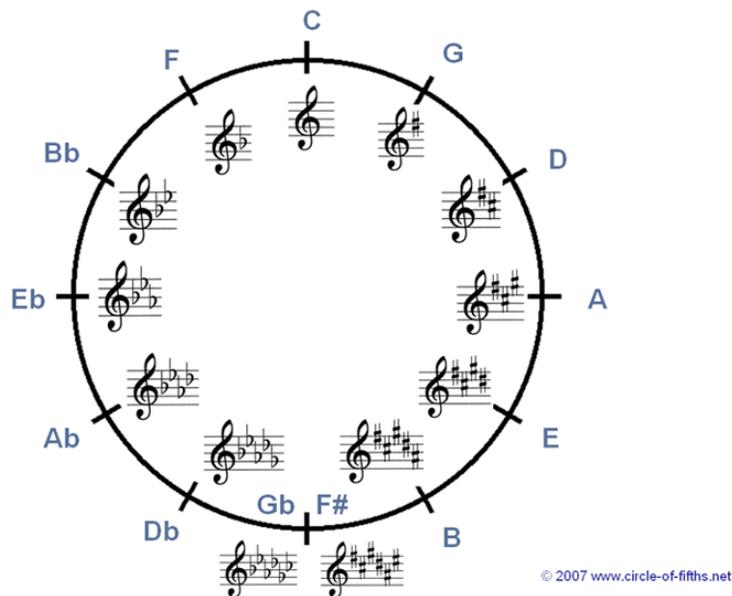

Figure i.
If A is represented by $y = sinx$, where x is the frequency, then B would be represented by $y = sin2x$, and C would be represented by $y = sin3x$.
http://www.marco-learningsystems.com/pages/sawyer/music.html
The first image is the full length of the string vibrating. The second is the string divided in two, which doubles the frequency and produces a tone an octave higher. The third is the string divided in three, which triples the frequency, and produces a tone another fifth higher.

What can provide barriers with musical notation of the tones from the series, however, is that as we go up the harmonic series to higher harmonics, the step size decreases. This happens because we add the fundamental frequency to obtain each new harmonic. Adding x to 2x makes a much bigger difference than adding x to 10x. So the step size is related to the ratio of frequency change each time the fundamental frequency is added. Since a whole and half step in music are set step sizes, we will eventually, as the intervals get smaller, have intervals from the harmonic series that are less than half steps. Music notation only allows for the fixed step size between adjacent notes, which are half steps. The awkward intervals of the upper harmonics, which are not whole or half steps, sound unfamiliar, and are often considered unpleasing. This is why we say the interval is pleasing if the frequencies are related by small integers; the requirement of *small* integers keeps us in the lower harmonics.

Harmonic series are unique to the fundamental tone. A very strong pure interval is the perfect fifth, which can be made into a circle. By a circle, we mean that if we start on a note, and the next addition to the circle is a fifth above that, then a fifth above that, and continue in this manner, this includes all of the notes in the chromatic scale.

Figure j. This is a diagram of the Circle of fifths. Begin on any note, and the notes to either side are a fifth above or below that note.

This is how music theorists organize relating keys. Related keys have similar key signatures, which define the sharps and flats in that key, are close to each other in the circle of fifths, and therefore are related by closely by fifths. If followed around a circle it will lead back to the original note. What's interesting is that the frequency found by following the circle of fifths around from one note will not be exactly the same as if followed around from another starting note. Look at the note D, a step above a C, for example. Based on the harmonic series of A, two steps below a C, it is a fourth above the fundamental, or a fourth above an A, so the frequency would be $\frac{4}{3}(220)$Hz $= 293.3$Hz. Based on the harmonic series of G three steps below a C, however, D is a fifth above the fundamental, a G, which means its frequency would be $\frac{3}{2}(200)$Hz $= 294$Hz. This difference may be small, but the fundamentals differ by only one whole step. It would be more amplified if the fundamentals were farther apart. If the circle of fifths based on the fundamental of A at 220Hz is taken all the way around to another A, the frequency of the new A would be $220\left(\frac{3}{2}\right)^{12} = 28544.2$Hz,

which should be a power-of-2 multiple of 220Hz, but is not. As this new A is seven octaves higher, the frequency, based on perfect octaves having frequency ratios of 2:1, should be $220(2^7) = 28160Hz$. Now the difference is more pronounced. This is called the Pythagorean Comma, and will lead us later to discussions of tuning systems.

6. THE FOURIER SERIES AND MODELING MUSIC

Now that we've covered the basics of the mathematics and the music, we can discuss the connection to the Fourier Series. As mentioned above, the wave equation can be used to model sound, as sounds are produced in waves. The Fourier Series is a solution to the wave equation, and can therefore be used to model sound. What is so convenient about the Fourier Series, is that it can break up the sound into pieces: trigonometric functions with frequencies and amplitudes. These functions are the fundamental and its harmonics. The fundamental can be represented as the first term after the constant A_0 in the Fourier Series solution for the wave (see Equation 2), which would be the first term in the sum, constituting a cosine and sine term. While frequencies can be represented with a single sinusoidal function, the form of the Fourier Series used for this paper has two functions for each term, which can form a single wave. Note that one of the terms may well be 0. The first non-constant term, which represents the fundamtental is:

$$A_1 cos\frac{\pi x}{a} + B_1 sin\frac{\pi x}{a}. \qquad\qquad \textbf{(4)}$$

Note that if the sound is a pure tone, the Fourier Series is one wave with one frequency, so the fundamental will be the only non-constant. The first harmonic can by represented as the second term in the sum in a Fourier Series:

$$A_2 cos\frac{2\pi x}{a} + B_2 sin\frac{2\pi x}{a}, \qquad\qquad \textbf{(5)}$$

and so on with the terms of the series, leaving the *nth* term as

$$A_n \cos \frac{n\pi x}{a} + B_n \sin \frac{n\pi x}{a}. \qquad \textbf{(6)}$$

The coefficients of the harmonics give the amplitude of each harmonic and determine the tonal quality, or timbre. Thus, the frequency of the fundamental tone determines the pitch we hear, and the harmonic contribution, as can be clearly seen in the Fourier Series, determines the timbre.

7. TUNING SYSTEMS

We've just discussed the ratios of the pure intervals that occur naturally in the harmonic series. But these exact ratios don't always apply in modern music. The above-mentioned uniqueness of each harmonic series and circle of fifths to its fundamental tone is the reason for this. If the composer wants to change keys, in order to have pure intervals, different frequencies are needed based on ratios from the new key. In addition, some notes from the upper harmonics will sound particularly dissonant, not to mention that we have no way of notating ever-decreasing step sizes. To solve this problem, various tuning systems have been developed throughout history for instruments.

The Pythagorean tuning system is based on the interval of a perfect fifth, and very closely on the harmonic series, using small integer ratios. This system fills in the chromatic scale with a series of fifths, as in the circle of fifths. In order to get to that final perfect octave, which is the seventh octave after the fundamental, eleven perfect fifths, and a significantly smaller fifth are needed. This sounds good when the not from smaller fifth is avoided, but doesn't work well otherwise. This system is a theoretical system that hasn't really been put into practice because of the problems of modulation and inconsistent fifths. Many historical systems have modified the Pythagorean system to keep some intervals pure, and some approximated, but many of these still had limits.

The current dominant system is called equal temperament. This system approximates pure, Pythagorean ratios, but in a way that allows modulation and consistency. In an octave, there are twelve chromatic steps, which are half steps. Instead of having these steps vary slightly as they go up the harmonic series, equal temperament divides the octave into twelve equal steps. Since the frequency ratio for an octave is 2:1, this means that each half step has a frequency of

$$u_n = u_0 2^{\frac{n}{12}}, \qquad \textbf{(7)}$$

where u_0 is the fundamental frequency and n is the number of half steps from the fundamental note. Conversely, the number of half steps n between two frequencies u_1 and u_2 is a logarithmic equation of base $2^{\frac{1}{12}}$,

$$n = \log\left(u_1/u_2\right). \qquad \textbf{(8)}$$

Since each half step is exactly the same size, a G in the key of D will be exactly the same frequency as a G in the key of A. This allows for modulation. Also, following the fifths up, after 12 fifths, we will end up on a perfect octave, so we have consistency in frequencies.

This system does still have a downside: because we use the number $2^{\frac{1}{12}}$ for a half step, which is an irrational number, the intervals are not rationals, so they're not pure to the harmonic series intervals. Since the intervals in equal temperament are not the pure rationals from the harmonic series, technically, they're not quite in tune with each other. Key intervals in western music, such as the major third, the perfect fourth, and the perfect fifth, are very close, but some, such as the seventh, are noticeably off. This works out since the seventh is naturally dissonant sounding. To explain this, we divide each half step into 100 cents (yet another frequency measure). So the frequency of a given tone based on the fundamental and the number of cents difference is

$$u_n = u_0 2^{\frac{n}{1200}}. \qquad \textbf{(9)}$$

Since n is now the number of cents, and there are 100 cents in a half step, the number of half steps has become $100n$. This is just a way to more closely express interval sizes based on the frequency.

An equal tempered fifth, which is seven half steps, should be exactly 700 cents. The perfect fifth with ratio of 3/2, however, is 702 cents, a very minimal difference. Another way to compare the frequencies would be to compute each pure ratio, and compare the decimal to the decimal irrational number from equal tempered system.

Take the perfect fourth: $\frac{4}{3} = 1.3333\ldots$ from the Pythagorean ratio, and $2^{\frac{5}{12}} = 1.3484\ldots$ from equal temperament.

We can see the difference in frequency ratios, but using cents gives a standardized comparison method. See the chart below for the comparison of all notes in the chromatic scale.

Interval	Pythagorean Ratio	Cents	Eq. Temp. Approximation	Cents
Minor 2nd	256/243 = 1.035	90	1.0595	100
Major 2nd	9/8 = 1.125	204	1.1225	200
Minor 3rd	32/27 = 1.185	294	1.1892	300
Major 3rd	81/64 = 1.266	408	1.26	400
Perfect 4th	4/3 = 1.333	498	1.335	500
Tritone	729/512 = 1.424	612	1.414	600
Perfect 5th	3/2 = 1.5	702	1.498	700
Minor 6th	128/81 = 1.58	792	1.587	800
Major 6th	27/16 = 1.688	906	1.682	900
Minor 7th	16/9 = 1.778	996	1.782	1000
Major 7th	243/128 = 1.898	1110	1.888	1100
Octave	2/1	1200	2	1200

Figure k. This chart shows the frequency ratios for the two tuning systems in numbers, and in cents, based on the interval.

This leads to the question of Fourier Series representations of the sounds produced by equal tempered instruments verses Pythagorean tuned instruments. If equally tempered instruments have slightly different intervals between steps, does that mean they have different harmonics? The equal temperament system is just a method to have consistent step sizes when the instrument changes notes. This can change the fundamental frequency of certain notes, but the physics of sound still apply, in that each tone still has harmonics that are integer multiples of the fundamental frequency. A Fourier Series of a tone will still clearly represent the fundamental and the overtones in the same way, whether the tone is from an equally tempered or Pythagorean tune instrument. If the fundamentals of two tones are different, then the Fourier Series will be different. This would be slight if the difference is only 2 cents, as in the case of perfect vs. tempered fifths.

8. PRACTICAL APPLICATIONS AND THE FOURIER TRANSFORM

While all this theory is interesting in itself, it does have some practical applications as well. Derived from the Fourier Series, the Fourier Transform can be used to turn musical signals into frequencies and amplitudes, which is what we need to understand it in terms of harmonics and the series. A simple form of the Fourier Transform, $F_k[f(x)](k)$, is given by

$$F_k[f(x)](k) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}dx . \qquad (10)$$

This can be derived from the generalized form of the Fourier Series and is the key to practically applying Fourier Analysis to music in audio form. This idea transforms our equations in the time domain to the frequency domain, or vice versa.

Dr. Jason Brown, a mathematician at Dalhousie University in Canada, put this to use recently on a popular Beatles song, *A Hard Day's Night*. To musicians, the opening chord (a distinct *chang*) has long been a mystery. Many scores of the song have tried to reproduce it, but none have ever quite matched. Brown decided to run the Fourier Transform on a one second recording of the chord using computer technology, and got a list of frequencies out—over 29,000 of them. He took only the frequencies with the highest amplitudes, as these would most likely be

the fundamental frequencies, and possibly some strong harmonics. He then compared these frequencies to an A of 220Hz, using the half step frequency change for equal tempered instruments as discussed above, and found how many half steps each was from the A. This was easily converted into a list of notes being played. Values of half steps that were not close to integers could be accounted for by out of tune instruments. Then, based on the instruments in the band and their physical capabilities, such as how many strings the piano and guitar instruments have for each note, he assigned each note to the instruments. The result was that he successfully recreated the chord as it was meant to sound, which others had only managed to approximate [Brown].

Another possible use of Fourier Analysis in music is in using it to compose the music. Computer generated 'spectral' music originated in Paris in the 1970's, and emphasizes timbre, not pitch and rhythm as in traditionally composed music. It focuses on the internal frequency spectrum of the sound. Composers use Fourier Analysis to see and change the timbre of the sounds they're creating. This could allow composers to create entirely new sounds, and not be confined by the physical capabilities of musical instruments.

9. CONCLUSIONS

As we can see, mathematics in music runs deep. The naturally pleasing ratios used in music are so pleasing because of the mathematical principals behind them, and all western music is based on the harmonic series. Modern tuning systems can be used to solve problems of modulation and consistency caused by the pure ratio intervals that our ears want to hear. Fourier Analysis is useful in modeling and breaking up sound, and the Fourier Transform opens up practical possibilities to model and define sound using Fourier Analysis.