The Superellipse Conjecture

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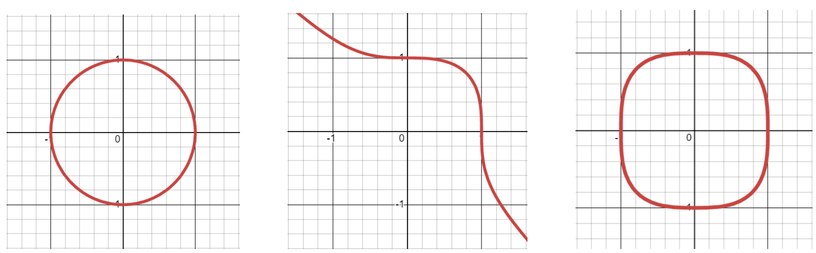
ABSTRACT
The focus of this research project will be to support what we call the **superellipse conjecture**, a mathematical claim about shapes called superellipses. Superellipses are also called “squircles” for their geometric relationship to squares and circles. The superellipse conjecture claims that a certain pattern is present among superellipses with particular areas. The project will collect data to support this conjecture through computer calculations, and will additionally try to find a mathematical proof for the claim.

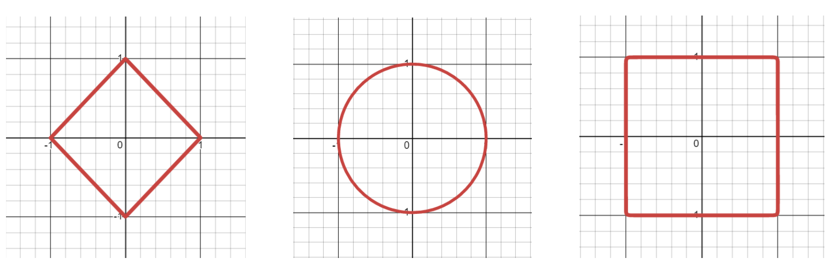
PROBLEM INTRODUCTION
In algebra, the unit circle has the following equation: $x^2 + y^2 = 1$. In this equation, the variables x and y are both being raised to the second power. The graph of this equation (a unit circle) is in Figure 1a. If we modify the equation to $x^3 + y^3 = 1$, the resulting graph no longer exhibits the same symmetries. Note that for any positive or negative number x, $x^2 = |x^2|$, i.e., the square of x is always the same as the absolute value of the square of x. This implies that the equation $x^2 + y^2 = 1$ is the same as the equation $|x^2| + |y^2| = 1$. If we use this fact to adjust the equation: $x^3 + y^3 = 1$ to: $|x^3|+ |y^3| = 1$, we obtain a graph (see Figure 1c.) that recovers some of the lost symmetries that did not appear in Figure 1b.



1a. $x^2 + y^2 = 1$ 1b. $x^3 + y^3 = 1$ 1c. $|x^3| + |y^3| = 1$

Figure 1: Graphs of related equations

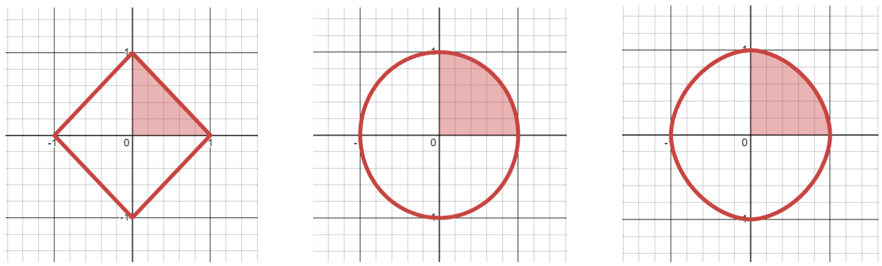
In Figure 1c, we have an equation of the form $|x^n|+ |y^n| = 1$ (in this graph, $n = 3$). This equation represents a unit superellipse. Note that when $n = 2$, the superellipse is just a unit circle (as seen in Figures 1a and 2c). When $n = 1$, the superellipse forms the square depicted in Figure 2a. As the exponent n grows, the superellipse inflates to a shape that slowly approaches a 2×2 square. As n grows, so does the area.



2a. $|x^1| + |y^1| = 1$ 2b. $|x^2| + |y^2| = 1$ 2c. $|x^{50}| + |y^{50}| = 1$

Figure 2: Graphs of $|x^n| + |y^n| = 1$ for various n

The area of the shape in Figure 2c is slightly less than 4. As n grows, the area of the corresponding superellipse approaches 4. Just as the area of the entire superellipse approaches 4, a quarter of its area, or its "quarter-area," approaches 1. Note that when $n = 1$, a quarter of the superellipse's area is equal to the area of the right triangle in quadrant 1 (see Figure 3a) which is exactly ½. In other words, as n grows larger and larger from 1, the area of the shaded regions shown in Figure 3 ranges from ½ to 1. There are famous infinite sequences of numbers which share this same property. One such example is the following sequence: {½, ¾, ⅞, …}. This is an infinite list of numbers which range from ½ to 1. Since the quarter-area of the superellipse also ranges from ½ to 1, it must be that each number in that sequence represents the quarter-area of a superellipse corresponding to some number n.



3a. $n = 1, A = ½$ 3b. $n = 2, A = \pi/4$ 3c. $n \approx 1.79, A \approx ¾$

Figure 3: Graphs with various quarter-areas

This was the inspiration for this project - to find what numbers n would correspond to areas in the form ½, ¾, ⅞, etc. The first n is already known, because $n = 1$ gives us the shape in Figure 3a whose quarter-area is ½. What if we want a quarter of our superellipse's area to instead be ¾? Recall that when $n = 2$, our superellipse is a unit circle. A unit circle is just a circle with radius 1, so its area is $\pi r^2 = \pi(1)^2 = \pi$. Recall that 3.14 > 3, and that this implies that $\pi/4 > 3/4$. Since our area grows with n, and $n = 2$ makes the quarter-area $\pi/4$, it must be that the next n in the sequence is less than 2. The superellipse whose quarter area is ¾ is obtained by calculation when n is approximately 1.79 (see Figure 3c). For ⅞, the appropriate n value was found to approximately be 2.87. We can make a sequence out of these n values. If we let $n_1 = 1$, $n_2 = 1.79$, $n_3 = 2.87$, etc., we will have a sequence of numbers where the k^{th} (k is a positive whole number like: 1, 2, 3, etc.) item of the sequence: n_k gives us a superellipse whose area is $(2^k - 1)/2^k$. Fifty terms of this sequence were obtained with a computer. The following trend was observed: as the sequence n_k continues and k grows larger, the ratio of one term over the previous: n_{k+1} / n_k approaches $\sqrt{2}$.

It was later found that this observed trend was a part of a larger, more general pattern. Recall that in constructing the sequence n_k, we had a defining requirement for our k^{th} term, i.e., that the k^{th} term would yield a quarter-area of $(2^k - 1)/2^k$. It turns out that the only special role played by the number 2 for this requirement is that it is a real number greater than 1. If we instead require that each n_k yield a quarter area of $(3^k - 1)/3^k$, we would subsequently find that the ratio n_{k+1} / n_k approaches $\sqrt{3}$. This brings us to the **superellipse conjecture:**

Fix r to be a real number greater than 1. Define $A(u) := \int_0^1 (1 - x^u)^{1/u}\,dx = [\Gamma(1 + 1/u)]^2 / \Gamma(1 + 2/u)$. For each positive integer k, define n_k to satisfy: $A(n_k) = (r^k - 1)/r^k = 1 - 1/r^k$. Then $\lim_{k\to\infty} n_{k+1} / n_k = \sqrt{r}$.

Note Γ is the Gamma function (see Davis 1959). Note that the expression $A(u)$ denotes the value of the aforesaid "quarter-area" we are interested in. Specifically, $A(u)$ denotes the quarter-area of the superellipse determined by the graph $|x^u| + |y^u| = 1$, (see Dirichlet 1839 and Wittaker, Watson 1996), since the region under the graph is bounded by $f(x) = (1-x^u)^{1/u}$ and the area under the graph of $f(x)$ in the first quadrant is given by $\int_0^1 f(x)\,dx$. The **superellipse conjecture** is an unproven mathematical statement (supported by computational evidence), and the goal of this research project is to provide a formal mathematical proof of why it is true.

METHODS

The **superellipse conjecture** is a novel mathematical statement we are trying to prove. Developing a formal proof involves theoretical analysis of the statement and the mathematical objects therein. A preliminary part of this analysis required data collection that supported the conjecture. The following R code was developed in order to collect data supporting the conjecture:

```
r=2.3
fn<-function(n,k){beta(1/n,1/n+1)/n-1+1/(r^k)}
```

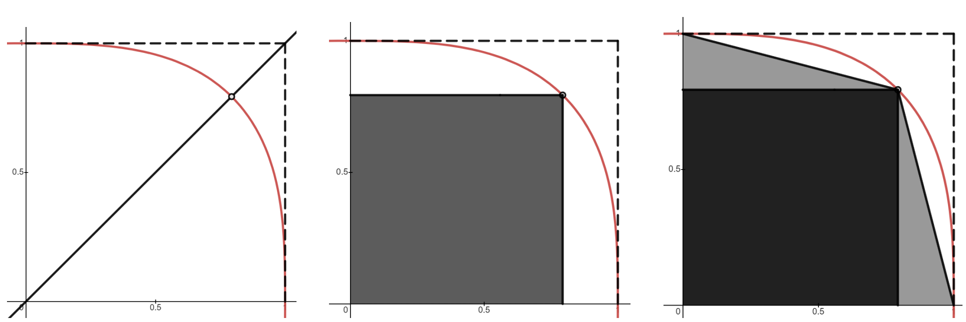
```
n.iter=100
lim=.0001 # The smaller this number is, the more precisely n is estimated.
dat=matrix(data=NA,nrow=n.iter,ncol=2)
dat[,1]<-1:n.iter
colnames(dat)=c("k","n_k")
# For k = 1 to k = n.iter, find the values of n that make the function fn equal to zero.
i=1
for(i in 1:n.iter){
  print(i)
  dat[i,2]<-uniroot(fn,lower=0,upper=1e100,f.lower=-lim,f.upper=lim,k=i)$root
  print(dat[i,2])
}
```

This code first sets a real number value to the parameter r (playing the role of *r* in the conjecture). Then, for each integer value of $1 \leq k \leq 100$, the code estimates the real number value of n which satisfies $[\Gamma(1 + 1/n)]^2 / \Gamma(1 + 2/n) - [1 - 1/r^k] = 0$. For each k, choosing n in this way finds the value of n so that the quarter area A(n) is as close to $1 - 1/r^k$ as possible, where A(u) is as defined earlier. The value of k and n are then stored to be plotted on a graph with the following code:

```
index<-8:20 # Indexing which values to plot
plot(dat[index,1],log(dat[index,2],base=r),xlab="k",ylab="logr(n_k)")
coef<-summary(lm(log(dat[index,2],base=r)~dat[index,1]))$coefficients
# This pulls out the coefficients of a simple linear regression of log(n_k) against k.
intercept<-coef[1,1]
slope<-coef[2,1]
abline(intercept,slope,col="red",lty=1) # This makes a red fitted line.
legend("topleft",legend=c(paste0("intercept=",intercept),paste0("slope=",slope)),bty="n")
# Creates the legend
```

The code above plots a graph wherein the x-axis is the k values, and the y-axis is $\log_r$(dat[index,2]), where dat[index,2] stores the value of n satisfying: $A(n) - [1 - 1/r^k] = 0$. Plotting the data in this way allows us to observe whether or not the pattern stated by the conjecture holds. According to the superellipse conjecture, the term-by-term growth seen in the n_k sequence is roughly exponential, which means it should be roughly log-linear. This is in fact what we observed for various values of r.

After obtaining this data, we geometrically analyzed the graph of the superellipse. The steps of this analysis are laid out in Figure 4, based on fixing a constant u. Recall that $f(x) = (1-x^u)^{1/u}$. We will look for a geometric region whose area is smaller than the quarter area A(u), the area below the red curve. Figure 4a depicts finding the value of x which equals $f(x)$, and the area of the square in Figure 4b is clearly a lower bound for the area below the red curve. A better lower bound is obtained by adding the two triangles in Figure 4c.

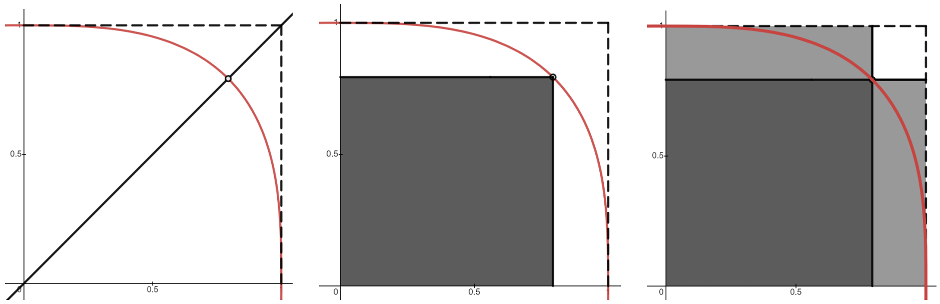


4a. Solve $x = f(x)$ 4b. Obtain square 4c. Add triangles

Figure 4: Finding a lower-bound area formula

Following these steps gives us a quadrilateral defined in terms of the constant u from $f(x)$. More specifically, the quadrilateral has vertices (0,0) and (1,0) and (0,1) and the point on the curve $f(x) = (1-x^u)^{1/u}$ where $x=f(x)$. This equation has a solution at $x = (½)^{1/u}$ for all u. This tells us that the quadrilateral's fourth corner is the point $((½)^{1/u}, (½)^{1/u})$, meaning the side length of the square in Figure 4b is $(½)^{1/u} – 0 = (½)^{1/u}$. Therefore, its area is $((½)^{1/u})^2 = (½)^{2/u}$. To find the area of the two additional triangles, we must find their base and height. If we look at the top triangle in figure 4c, we can see that its base is the side length of the square: $(½)^{1/u}$, and its height is simply $(1 - (½)^{1/u})$. So the total area of the two triangles is

$2*\frac{1}{2}*(\frac{1}{2})^{1/u}*(1-(\frac{1}{2})^{1/u}) = (\frac{1}{2})^{1/u} - (\frac{1}{2})^{2/u}$. The area of the quadrilateral is simply the area of the square in Figure 4b with the area of the two triangles in Figure 4c added on. Therefore, its area is given by $A_{lb}(u) = (\frac{1}{2})^{2/u} + [(\frac{1}{2})^{1/u} - (\frac{1}{2})^{2/u}] = (\frac{1}{2})^{1/u}$. Based on the picture, we know that the area of this quadrilateral is less than the quarter-area of the superellipse. To find a shape whose area is greater than the quarter area, a similar analysis was performed. The steps are laid out in Figure 5.



5a. Solve $x = f(x)$ 5b. Obtain square 5c. Add rectangles

Figure 5: Finding an upper-bound area formula

The area of the shaded region in Figure 5b is simply the total area of the 1×1 square minus the area of the empty region in the top-right corner of the square in Figure 5c. This is given by: $A_{ub}(u) = 1 - (1 - (\frac{1}{2})^{1/u})^2 = -(\frac{1}{2})^{2/u} + 2(\frac{1}{2})^{1/u}$. Again, based on the picture, we can see that this region has an area that is greater than the quarter area. Using these area formulas: $A_{lb}(u)$, $A_{ub}(u)$, we can construct new sequences. Fix $r > 1$, and for each positive integer k, define N_k to satisfy $A_{lb}(N_k) = 1 - 1/r^k$, and define M_k to satisfy: $A_{ub}(M_k) = 1 - 1/r^k$. Unlike our original area formula, we can algebraically solve for when these requirements are met.

First, we show that $N_k = -1 / \log_2(1-1/r^k)$. Suppose $A_{lb}(w) = (\frac{1}{2})^{1/w} = 1 - 1/r^k$. Now we solve for w.

$\log_2((\frac{1}{2})^{1/w}) = \log_2(1 - 1/r^k)$.

$\log_2((2^{-1})^{1/w}) = \log_2(1 - 1/r^k)$.

$1/w * \log_2(2^{-1}) = \log_2(1 - 1/r^k)$.

$1/w * (-1) = \log_2(1 - 1/r^k)$.

$1/w = (-1) * \log_2(1 - 1/r^k)$.

$[1/w]^{-1} = [-\log_2(1 - 1/r^k)]^{-1}$.

$w = -1 / \log_2(1-1/r^k)$. Therefore, if $A_{lb}(N_k) = 1 - 1/r^k$, then $N_k = w = -1 / \log_2(1-1/r^k)$.

We proceed in showing that $M_k = -1 / \log_2(1-1/r^{k/2})$. Suppose $A_{lb}(w) = -(\frac{1}{2})^{2/w} + 2(\frac{1}{2})^{1/w} = 1 - 1/r^k$. Let $x = (\frac{1}{2})^{1/w}$. Our equation then becomes: $-x^2 + 2x - (1 - 1/r^k) = 0$. Using the quadratic formula with $a = -1$, $b = 2$, and $c = -(1 - 1/r^k)$, we find that $x = 1 \pm 1/r^{k/2}$. Substituting $(\frac{1}{2})^{1/w}$ for x, we must now solve $(\frac{1}{2})^{1/w} = 1 \pm 1/r^{k/2}$.

$\log_2((\frac{1}{2})^{1/w}) = \log_2(1 \pm 1/r^{k/2})$.

$1/w * (-1) = \log_2(1 \pm 1/r^{k/2})$.

$w = -1 / \log_2(1 \pm 1/r^{k/2})$.

Both of these are valid solutions, but if $M_k = -1/\log_2(1 + 1/r^{k/2})$, then M_{k+1} / M_k turns out to be undefined. This is because $[-1/\log_2(1 + 1/r^{(k+1)/2})] / [-1/\log_2(1 + 1/r^{k/2})] = \log_2(1 + 1/r^{k/2}) / \log_2(1 + 1/r^{(k+1)/2}) = \log_2((1 + 1/r^{k/2}) - (1 + 1/r^{(k+1)/2})) = \log_2(1/r^{k/2} - 1/r^{(k+1)/2})$. Since $r > 1$, this final expression we get is plugging a negative number into a logarithm, and so this expression is undefined. Hence $M_k = -1/\log_2(1 - 1/r^{k/2})$ as the solution we used for computing ratios.

Using R code, we found that $\lim_{k\to\infty} N_{k+1} / N_k = r$, and $\lim_{k\to\infty} M_{k+1} / M_k = \sqrt{r}$.

The motivation behind this analysis via lower and upper bounds is as follows: for each u, we have $A_{lb}(u) \leq A(u) \leq A_{ub}(u)$. Recall for each positive integer k, we had n_k was defined to satisfy $A(n_k) = (r^k - 1)/r^k = 1 - 1/r^k$, and motivated by this, N_k and M_k were analogously defined to satisfy $A_{lb}(N_k) = 1 - 1/r^k$ and $A_{ub}(M_k) = 1 - 1/r^k$. Since $A_{lb}(u) \leq A(u) \leq A_{ub}(u)$ holds, we investigated $\lim_{k\to\infty} N_{k+1} / N_k$ and $\lim_{k\to\infty} M_{k+1} / M_k$, and also asked whether either of these limits, if their values were obtained, would be relevant in determining $\lim_{k\to\infty} n_{k+1} / n_k$, the limit introduced in the **superellipse conjecture**.

Other methods to analyze the conjecture were also used. One such method attempted to reverse the process of the construction of the sequence n_k. Note that we ended with the claim that $\lim_{k\to\infty} n_{k+1} / n_k = \sqrt{r}$. To reverse this process, we instead start by defining a sequence m_k such that $m_k = (\sqrt{r})^{k-1}$. Using the same area formula for $A(u)$, it is possible to compute values of $A(m_k)$. Note that for the sequence n_k, by definition: $A(n_{k+1}) - A(n_k) = (r^{k+1} - 1)/r^{k+1} - (r^k - 1)/r^k = [(r^{k+1}$

$- 1) - r(r^k - 1)]/r^{k+1} = [r - 1]/r^{k+1}$. In other words, our *roughly* exponentially-growing n_k inputs are producing outputs that grow *exactly* at the rate of a geometric series. We felt it was valuable to investigate if our *exactly* exponentially-growing m_k inputs would produce outputs that grow *roughly* at the rate of a geometric series. Put more precisely, if l_k satisfies: $A(m_k) = 1-1/r^{l_k}$, then we expect this sequence to grow such that $\lim_{k\to\infty} l_{k+1} - l_k = 1$. Note that $l_{k+1} - l_k$ is given by the expression: $\log_r(1/1-A(m_{k+1})) - \log_r(1/1-A(m_k))$. To see if this claim about the sequence l_k is true, the following R code was written:

```
k=1:100
# Index of sequence
r=1.5
# Value of r
m_k<-sqrt(r)^(k-1)
# Defining m_k
x<-seq(0,1,length.out=10000000)
x.interval<-x[2]-x[1]
A.matrix<-matrix(data=NA,nrow=length(k),ncol=2)
# Creates a 2-column, k-row matrix
colnames(A.matrix)<-c("k","l_k")
# Names columns of matrix
options(digits = 22)
# Changes number of decimal places stored
for(i in 1:length(k)){
  print(i)
  A.matrix[i,1]<-i
  # Stores value of i [i: 1->k] as first-column, kth row entry of matrix
  A.matrix[i,2]<-log(1/(1-((gamma(1+1/m_k[i]))^2 / gamma(1+2/m_k[i]))),base=r)
  # Stores l_k value as second-column, kth row entry of matrix
}
A.matrix[2:100,2]-A.matrix[1:99,2]
# Finds values of l_{k+1}-l_{k}
```

This code first specifies an integer value for the parameter k, signifying how many terms of the sequence l_k we are collecting. It then assigns a real number number value to the parameter r. It then creates a matrix wherein the value of i is stored in the first column and the value of $l_{i+1} - l_i$ is stored in the second column. Using this code, we were able to track the value of $l_{k+1} - l_k$. Additionally, we wrote some R code to plot the sequence l_k against its index k.

RESULTS AND DISCUSSION

Figure 6 below displays some of the data for the sequence n_k obtained from R code.

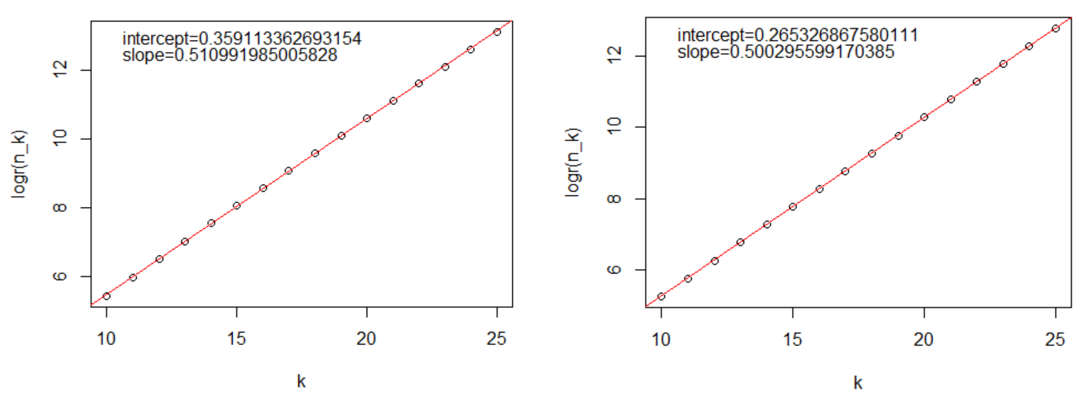


Figure 6a. r = 1.5 Figure 6b. r = 2.5

Figure 6: R Data for n_k sequence

The data shown in Figure 6 supports the **superellipse conjecture**. This is because these (for k) graphs show a log-linear relationship between the values of k and n_k. Moreover, since the slope of the graph is roughly ½, and the logarithm's base is r, the data supports the conjecture's claim that from n_k to n_{k+1}, there is roughly a proportional growth of $r^{1/2}$, which is the same as $\sqrt{r}$. Apart from 1.5 and 2.5, several other values of r were tested with the R code, and similar results were obtained in each case.

Figures 7 and 8 below display data from the sequences N_k and M_k, respectively.

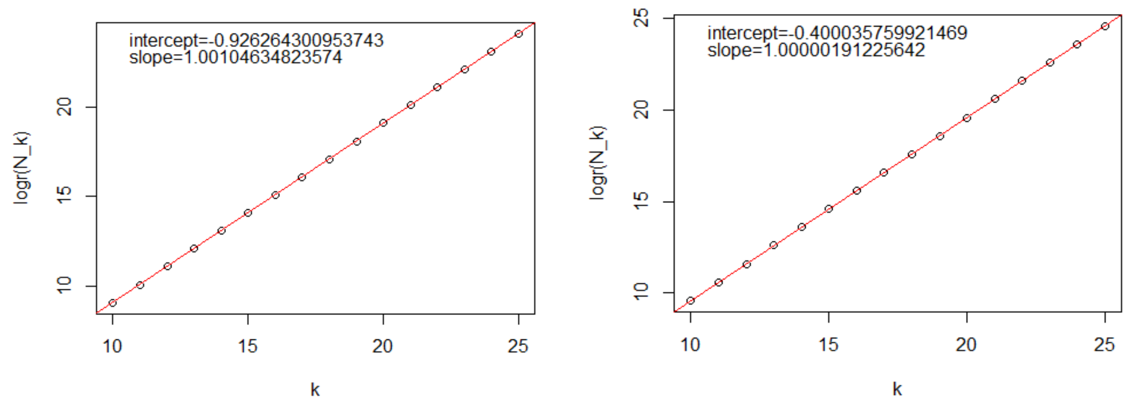


Figure 7a. r = 1.5 Figure 7b. r = 2.5

Figure 7: R Data for sequence N_k

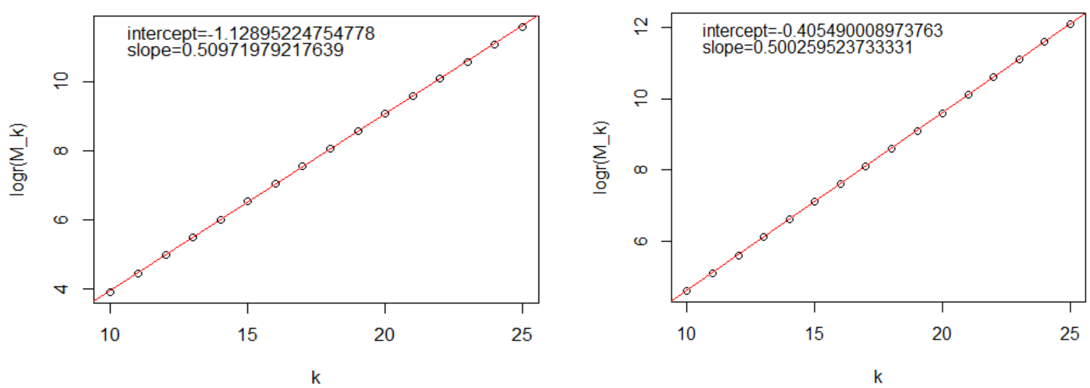


Figure 8a. r = 1.5 Figure 8b. r = 2.5

Figure 8: R Data for sequence M_k

Figure 7 above shows data supporting our claim that $\lim_{k\to\infty} N_{k+1} / N_k = r$. Similarly, Figure 8 shows data supporting our claim that $\lim_{k\to\infty} M_{k+1} / M_k = \sqrt{r}$. In terms of proving the **superellipse conjecture**, the behavior of the sequence M_k is more promising. In order to prove that the conjecture is true, we want to say something about the behavior of the sequence n_k. The behavior of the sequence n_k is determined by two things: the area formula $A(u)$, and the requirement that $A(n_k) = 1 - 1/r^k$. By constructing $A_{lb}(u)$ and $A_{ub}(u)$ and fixing the second requirement, we had hoped to develop a "squeeze" proof. Roughly speaking, the proof would obtain an upper bound formula $F_{ub}(u)$ *and* a lower bound formula $F_{lb}(u)$ such that sequences using these area formulas would have the same behavior as the sequence n_k. The proof would then need to show that because $F_{lb}(u) \leq A(u) \leq F_{ub}(u)$, and the resulting sequences each have limit ratios of $\sqrt{r}$, it follows that n_k, a sequence constructed from an area formula squeezed between the other two would have to have this same limit ratio.

We were unable to construct a satisfactory lower bound $F_{lb}(u)$ formula. Our best attempt, $A_{lb}(u)$, produced the sequence N_k, and we found that $\lim_{k\to\infty} N_{k+1} / N_k = r$. As such, we were not able to construct a formal proof of the conjecture in this desired manner. That is not to say this approach to proving the conjecture is useless. Recall that the $A_{ub}(u)$ formula produced the sequence M_k, which displayed our desired rate of growth. To continue with this method of proof, future research on this conjecture will need to construct a satisfactory $F_{lb}(u)$.

Figure 9 shows data collected from our other approach: reversing the process. For $r = 1.5, 2$, it plots the sequence l_k against each integer k, which was obtained from R code which was nearly identical to code for plotting presented earlier.

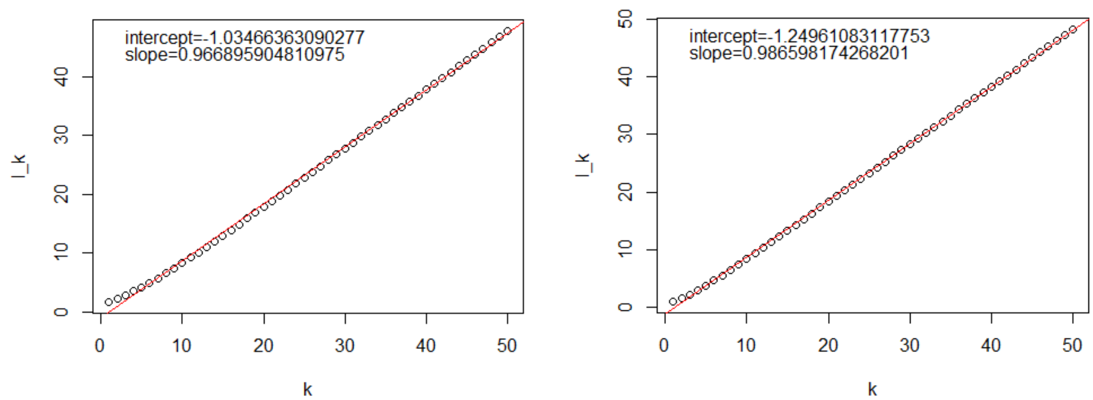


Figure 9a. $r = 1.5$ Figure 9b. $r = 2$

Figure 9: R Data for the sequence l_k

As the plots in Figure 9 show, l_k was found to have a roughly linear relationship with the index value k. This data supported our claim about the sequences l_k and m_k. When m_k grows at an exactly exponential rate, l_k has a roughly linear relationship with the index k. Although we did not obtain a formal proof that this relationship holds, doing so may prove useful in future research into the conjecture.

LIMITATIONS

The most significant limitation in analyzing this conjecture is its involvement with the gamma function Γ. The gamma function has no closed form representation. This makes it extremely difficult to extrapolate any useful information from an equation involving the gamma function. Since such an equation is what defines the sequence n_k in the **superellipse conjecture**, circumventing the obstacle of computing with the gamma function is crucial in obtaining a proof. This is precisely what motivated our different approaches. Due to the difficulty of directly obtaining information about the sequence n_k, we constructed related sequences N_k, M_k, and m_k so that we could study this problem in other ways.

In addition to the theoretical difficulties introduced by the gamma function, there were also some computational difficulties in collecting data. This was especially problematic when obtaining data for the sequences n_k and l_k. There were two things contributing to this issue. One contributing factor is the function $A(u)$. This function maps arbitrarily large real numbers to the interval $(½,1)$. Since the formula for $A(u)$ uses the gamma function, its outputs can only be approximated. If these approximations are not sufficiently precise, round-off errors can start to occur. This brings us to the other factor: R's decimal precision. R can usually only store values to 22 decimal places of precision. This was usually sufficient for accurately calculating some terms of the sequence, but roundoff errors eventually occurred. In testing several distinct values for the parameter r, we found that the greater the value of r, the sooner the sequences would display these errors. Future projects on this conjecture can mitigate this computational limitation by writing similar code to calculate the sequences n_k and l_k using software that utilizes higher precision.

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REFERENCES

Davis, P. J. (1959). "Leonhard Euler's Integral: A Historical Profile of the Gamma Function". *American Mathematical Monthly*. 66(10):849–869. doi:10.2307/2309786. JSTOR 2309786.

Dirichlet, P. G. L. (1839). *Über eine neue Methode zur Bestimmung vielfacher Integrale*. Collected Works 1:389.

Whittaker, E. T. and Watson, G. N. (1996). *A course of Modern Analysis*. Cambridge University Press. (4th ed.).