The First Calculus Competition of UW-La Crosse
Date: 11/3/12

Name ___________________________ Score ________

Directions: Each problem is worth 10 points. Please show your work on the answer sheets to get full credit.

1. Calculate \( \lim_{x \to 0} \left(1 + x\right)^{\frac{1}{\sin x}} \).

2. Calculate \( \lim_{x \to 0} \left(\frac{1 + x}{\sin x} - \frac{1}{x}\right) \).

3. If \( f(x) = (e^x - 1)(e^{2x} - 2)(e^{3x} - 3) \cdots (e^{nx} - n) \), where \( n \) is a positive integer, find \( f'(0) \).

4. If \( f(x) = \begin{cases} \sin[a(x - 1)] & \text{if } x \leq 1 \\ \ln x + b & \text{if } x \geq 1 \end{cases} \), find the values of \( a \) and \( b \) such that \( f(x) \) is differentiable at \( x = 1 \).

5. Given implicit function \( y = y(x) \) determined by \( x^2 - y + 1 = e^y \), find \( \frac{dy}{dx} \).

6. Prove that \( \left(1 + \frac{1}{x}\right)^x < e \) for any \( x > 0 \).

7. The function \( f(x) \) is continuous on \([0, 3]\) and differentiable on \((0, 3)\). If \( f(0) + f(1) + f(2) = 3 \) and \( f(3) = 1 \), prove that there must exist \( \xi \in (0, 3) \) such that \( f'(\xi) = 0 \).

8. Calculate \( \lim_{n \to \infty} n \left(\frac{1}{1 + n^2} + \frac{1}{2^2 + n^2} + \cdots + \frac{1}{n^2 + n^2}\right) \).

9. Calculate the indefinite integrals \( \int \left(1 - \frac{1}{x^2}\right) \sqrt{x} \sqrt{x} \, dx \) and \( \int \frac{x}{1 + x^4} \, dx \).

10. Calculate the definite integral \( \int_{0}^{2} x \sqrt{2x - x^2} \, dx \).